

On the edge Roman domination in graphs ^{*†}

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Abstract

A *Roman domination function* on a graph $G = (V(G), E(G))$ is a labeling $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex with label 0 has at least a neighbor with label 2. *Roman domination number* $\gamma_R(G)$ of G is the minimum of $\sum_{v \in V(G)} f(v)$ over such functions. An *edge Roman domination function* of G is Roman domination function on $L(G)$ and *edge Roman domination number* of G , $\gamma_{LR}(G)$, is defined $\gamma_R(L(G))$. In this paper, we introduce some results on the edge Roman domination number of a graph. Also, we provide some upper and lower bounds for the edge Roman domination number of graphs.

1 Introduction

Throughout this paper, we will consider only simple graphs. Let G be a graph. The set of vertices and edges of G are denoted by $V(G)$ and $E(G)$, respectively. The *order* of G denotes the number of vertices of G . We denote the number of edges of G by $\varepsilon(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *closed neighborhood* of v denoted by $N[v]$ is defined as $N(v) \cup \{v\}$. Also, for every set $S \subseteq V(G)$, $N[S] = \cup_{v \in S} N[v]$. Similarly, for an edge $e \in E(G)$, we define $N(e) = \{f \in E(G) \mid e \text{ is adjacent to } f\}$ and $N[e] = N(e) \cup \{e\}$. For every set $S \subseteq V(G)$ a subgraph whose vertex set is S and whose edge set is the set of those edges of G that have both ends in S is an *induced* subgraph by S and is denoted by $G[S]$ or simply $[S]$. Similarly, for every set $S' \subseteq E(G)$ the *edge-induced* subgraph by S' is defined $G[V[S']]$ and denoted by $G[S']$ or simply $[S']$, where $V[S']$ is the set of vertices of S .

The degree of a vertex v in G is denoted by $d_G(v)$ or simply $d(v)$. The maximum degree of vertices of G is denoted by $\Delta(G)$ or simply Δ and the

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minimum degree is denoted by $\delta(G)$ or simply δ . In a tree T , a *leaf* is a vertex of degree 1 and the number of leaves in T is denoted by $l(T)$. Also, a *penultimate* vertex is a vertex adjacent to at least one leaf.

The *diameter* of G , denoted by $\text{diam}(G)$, is the maximum distance between vertices of G . The path and the cycle of order n are denoted by P_n and C_n , respectively. The Cartesian graph product $G \times H$ of graphs G and H with disjoint vertex sets, is the graph with vertex set $V(G) \times V(H)$ and vertex (v_1, v_2) adjacent to (u_1, u_2) , whenever $v_1 = u_1$ and $v_2 u_2 \in E(H)$, or $v_2 = u_2$ and $v_1 u_1 \in E(G)$. A *d-dimensional hypercube* Q_n , is defined recursively in terms of Cartesian product of two graphs as $Q_1 = K_2$ and $Q_{n+1} = Q_n \times K_2$.

A set $M \subseteq E(G)$ is called *matching* if no two edges in M are adjacent. A matching M is maximal if for every $e \in E(G) \setminus M$, $M \cup \{e\}$ is not a matching. We denote the size of the smallest maximal matching of G by $m(G)$.

A set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V(G)$. The *dominating number* of G , $\gamma(G)$, is the minimum cardinality of a dominating set in G and a dominating set S of minimum cardinality is called a γ -set of G . A set $S \subseteq V(G)$ is called *independent* if no two vertices in S are adjacent. A *Roman dominating function* of a graph $G = (V(G), E(G))$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. Let (V_0, V_1, V_2) be the ordered partition induced by f , where $V_i = \{v \in V(G) : f(v) = i\}$, so one can show a Roman dominating function by an ordered partition (V_1, V_2, V_3) . The weight of a Roman dominating function of G is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The minimum weight of a Roman dominating function of G is called the *Roman dominating number* of G and is denoted by $\gamma_R(G)$. The corresponding dominating function is called γ_R -function [1].

An *edge Roman dominating function (ERDF)* of a graph $G = (V(G), E(G))$ is a function $f : E(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every edge e for which $f(e) = 0$ is adjacent to at least one edge h for which $f(h) = 2$. Let $(E_0(G), E_1(G), E_2(G))$ be the ordered partition induced by f , where $E_i(G) = \{e \in E(G) : f(e) = i\}$. Similarly, an ordered partition $(E_0(G), E_1(G), E_2(G))$ of edges determines a unique ERDF. The weight of an ERDF f of G is the value $f(E(G)) = \sum_{e \in E(G)} f(e)$. The minimum weight of ERDFs of G is called the *edge Roman dominating number* of G and is denoted by $\gamma_{LR}(G)$. Consider a graph G and a function $f : E(G) \rightarrow \{0, 1, 2\}$. We call a vertex $v \in V(G)$ a *2-covered vertex*, if there is an edge $uv \in E(G)$ such that $f(uv) = 2$.

2 Previous works and our contribution

Stewart first introduced the notion of Roman domination in year 1999 in [9]. Later in 2000, Cockayne et al. studied many properties of Roman domination in graphs. In [1], they found upper bounds and lower bounds for $\gamma_R(G)$ in terms of $\gamma(G)$ in general graphs and some particular class of graphs such as trees, cycles and grids. Also, in [2], Favaran et al. have given upper and lower bounds for $\gamma_R(G)$ and $|V_i|$, when we have a Roman dominating function (V_0, V_1, V_2) .

Some other versions of Roman dominating functions are defined and studied too. Henning et al. studied weak Roman dominating functions in [3] and k -Roman dominating functions in [4]. K. Kammerling and L. Volkmann studied k -Roman dominating functions too, see [5].

Some authors have studied Roman domination functions in graphs under some conditions. In [11], Xing et. al. gave characterization of graphs in which $\gamma_R(G) = \gamma(G) + k$ for some integer k and in [8], the authors have studied Roman dominating functions in which each vertex is coevered by exactly on vertex. A graph G is called a *Roman graph* when $\gamma_R(G) = 2\gamma(G)$. In other words, one can find a minimum dominating function for G using only labels 2. Xuelling et al. found some classes of graphs which are Roman, see [12].

The computation of Roman dominating function is another interesting challenge. In [6], the authors gave a linear time algorithm for finding Roman domination number in interval graphs and other polynomial time algorithms to find Roman domination number of special classes of graphs.

Roushini Leely Pushpam and Malini Mai initiated the study of edge Roman domination in graphs. They found some properties of edge Roman dominating functions and found the edge Roman dominating number of paths and cycles in [7].

In this paper, we study the edge Roman dominating functions. First, we show that for a graph G of order n , $\gamma_{RL}(G) \leq \frac{2\Delta}{2\Delta+1}n$. Then we prove that edge Roman dmination number of a tree T of order n is at least $\lceil \frac{2(n-l(T)+1)}{3} \rceil$ and at most $\lceil \frac{2(n-1)}{3} \rceil$. In last section, we give some upper bound and lower bounds for the edge Roman domination number of some classes of graphs such as grids and hypercubes.

3 Some Preliminary Results

Theorem 1. [1] *Let $f = (V_0, V_1, V_2)$ be any γ_R -function. Then*

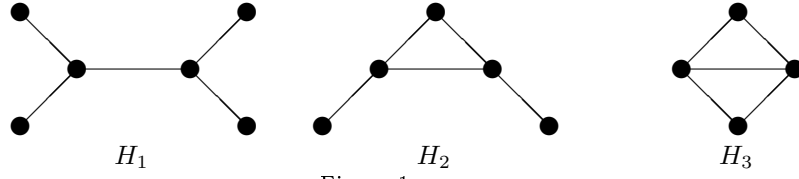
- (a) $G[V_1]$, the subgraph induced by V_1 has maximum degree 1.
- (b) No edge of G joins V_1 and V_2 .
- (c) Each vertex of V_0 is adjacent to at most two vertices of V_1 .
- (d) V_2 is a γ -set of $G[V_0 \cup V_2]$.

Theorem 2. [1] *Let $f = (V_0, V_1, V_2)$ be a γ_R -function of an isolated-free graph G , such that $|V_1|$ is a minimum. Then*

- (a) V_1 is independent.
- (b) Each vertex of V_0 is adjacent to at most one vertex of V_1 .

Lemma 3. *for every graph G there exists a minimum γ_{LR} -function f in which all edges with non-zero values form a matching.*

Proof. By Theorem ?, we can assume that no edge with value 1 is incident with edges with value 1 or 2. Suppose that $x, y, z \in V(G)$ and $f(xy) = f(xz) = 2$. If every vertex in $N(z)$ is 2-covered, then by changing $f(xz)$ to 0 we obtain an ERDF whose weight is less than f , a contradiction. Thus, assume that there exists $t \in N(z)$ such that t is not 2-covered. Now, change $f(xz)$ and $f(zt)$ to 0 and 2, respectively. In the new function the number of pairs of incident edges with value 2 decreases by at least 1. By continuing this procedure, we obtain a γ_{LR} -function in which all edges with non-zero value form a matching. \square



Theorem 4. *Let G be a graph of size m . If $\Delta(G) \leq 3$, then $\frac{2m}{5} \leq \gamma_{LR}(G)$. The equality holds if and only if G is a Roman edge graph and G is decomposed into some copies of H_1, H_2 and H_3 .*

Proof. Suppose that $f = (E_0, E_1, E_2)$ is a γ_{LR} -function for G . Since $\Delta(G) \leq 3$, $|N_G[e]| \leq 5$, for every $e \in E(G)$ and the equality holds if and only if both vertices of e are of degree 3. Therefore

$$m - |E_1| \leq \sum_{e \in E_2} |N(e)| \leq 5|E_2|,$$

and consequently

$$\frac{2m + 3|E_1|}{5} \leq 2|E_2| + |E_1| = \gamma_{LR}(G).$$

If the equality holds, then $E_1 = \emptyset$, $|N_G[e]| = 5$ for every $e \in E_2$ and $N[e] \cap N[e'] = \emptyset$, for every two distinct edges in E_2 . So G is a Roman edge graph and also by the above argument $G[N[e]]$ is a copy of H_1, H_2 or H_3 and then G is decomposed into some copies of H_1, H_2 and H_3 . \square

Theorem 5. *Let G be a graph of size m . If G is decomposable into some copies of H_1, H_2 and H_3 and $\Delta(G) \leq 3$, then G is a Roman edge graph and moreover $\gamma_{LR}(G) = \frac{2m}{5}$.*

Proof. We prove the assertion by induction on m . For $m = 5$ the assertion is trivial. If $5 \nmid m$, then we are done. Now, suppose that $5 \mid m$ and G is decomposable into r copies of H_1, H_2 and H_3 and $\Delta(G) \leq 3$. If G_1 is one of these copies, by the induction hypothesis $G \setminus E(G_1)$ is a Roman edge graph and $\gamma_{LR}(G \setminus E(G_1)) = \frac{2(m-5)}{5} = 2(r-1)$. Let $f = (E_0, E_1, E_2)$ be a γ_{LR} -function for $G \setminus E(G_1)$. We extend f by substituting the middle edge of G_1

by 2 and four other edges by 0. The new function f is a *RDF*-function for G and $\gamma_{LR}(G) + 2 \geq \gamma_{LR}(G \setminus E(G_1))$. On the other hand since $\Delta(G) \leq 3$, $G \setminus E(G_1)$ is an induced subgraph of G and by Theorem ?? we find, $\gamma_{LR}(G) \geq \gamma_{LR}(G \setminus E(G_1))$. Therefore, there are three cases for $\gamma_{LR}(G)$:

- $\gamma_{LR}(G) = \gamma_{LR}(G \setminus E(G_1))$. Since $\gamma_{LR}(G \setminus E(G_1)) = 2r - 2$, by the pigeon hole principle in one of the copies of H_i the induced value of f is less than 2. If H is this copy, then $f(E(H)) = 1$ and the unique edge with label 1 is the middle edge of H . By the restriction of f to $G \setminus H$ we obtain an *ERDF*-function for $G \setminus H$. But, $f(G \setminus H) = 2r - 3$ and a contradiction.
- $\gamma_{LR}(G) = \gamma_{LR}(G \setminus E(G_1)) + 1$. Similar to the previous case, there is a copies of H_i, H , such that the induced value of f to H is 1 and restriction of f to $G \setminus H$ is an *RDF*-function for it. In fact, f is a γ_{LR} -function for $G \setminus H$. Therefore we find:

$$\begin{aligned} 5(r-1) &\leq 5|E_2(G \setminus H)| + |E_1(G \setminus H)| \\ &\leq 3|E_2(G \setminus H)| + 2r - 2. \end{aligned}$$

Consequently, $|E_2(G \setminus H)| \geq r - 1$ and so $|E_1(G \setminus H)| = 0$ and $|E_2(G \setminus H)| = r - 1$. By induction hypothesis and Theorem 4 only middle edges of copies of H_i in $G \setminus H$ have label 2. Since $\Delta(G) \leq 3$, there is no edge with label 2 such that it is connected to the edges of H with label 0, a contradiction.

- $\gamma_{LR}(G) = \gamma_{LR}(G \setminus E(G_1)) + 2$. We claim that the induced values of f to the copies of H_i s are exactly 2. Otherwise, there is a copy, H , with middle edges value 1 and four other has value 0. Induced value of f to $G \setminus H$ has value $2r - 1$. Therefore $2r - 1 = 2|E_2(G \setminus H)| + |E_1(G \setminus H)|$. On the other hand

$$\begin{aligned} 5(r-1) &\leq 5|E_2(G \setminus H)| + |E_1(G \setminus H)| \\ &\leq 3|E_2(G \setminus H)| + 2r - 1. \end{aligned}$$

Consequently, $|E_2(G \setminus H)| \geq r - 1$ and therefore $|E_1(G \setminus H)| = 1$ and $|E_2(G \setminus H)| = r - 1$. So $|E_1| = 2$, $|E_2| = r - 1$ and $5r \leq 2 + 5(r - 1)$, a contradiction. Hence, $\gamma_{LR}(G) = 2r = \frac{2m}{5}$ and if in a copy of H_i an edge has label 1, exactly one of its neighbor edges has label 1 too, and by changing one of them to 2 and the other to 0, a γ_{LR} -function is obtained and G is a Roman edge graph.

□

4 Edge Roman domination of graphs

In the following theorem, we prove that if a graph G of order n and the maximum degree Δ has a perfect matching, then $\gamma_{LR}(G) \leq \frac{2\Delta-1}{2\Delta}n$. As a consequence of this theorem, we conclude that the edge Roman domination number of every bipartite k -regular graph is at most $\frac{2k-1}{2k}n$.

Theorem 6. *Let G be a graph of order n with maximum degree Δ . If G has a perfect matching, then $\gamma_{LR}(G) \leq \frac{2\Delta-1}{2\Delta}n$.*

Proof. Let M be a perfect matching. Define an ERDF of G , say g , $g(e) = 2$ for every $e \in M$ and $g(e) = 0$, otherwise. Suppose that $A \subseteq M$ is a maximal subset of edges of M such that the vertices of every two different edges of A are not adjacent in G . Change $g(e)$ to 1, for every $e \in A$ and keep the values of other edges. Clearly, g is still an ERDF. If $|V(\langle A \rangle)| \geq \frac{n}{\Delta}$, then $w(g) \leq \frac{2\Delta-1}{2\Delta}n$, and we are done.

Thus, suppose that $|V(\langle A \rangle)| < \frac{n}{\Delta}$. Assume that $B \subseteq M$ is the set of all edges of M such that for every edge $e \in B$, exactly one of the end points of e is adjacent to exactly one vertex of $\langle A \rangle$. Let $C = M \setminus (A \cup B)$. Since A is maximal, by the definition of B , for each edge e of C , the end points of e have totally at least 2 neighbors in A .

We claim that one can reduce $w(g)$ by $\frac{|B|}{2\Delta-1}$. Let $B' \subseteq B$. First, let $B' = \emptyset$. Continue the following procedure until $B' = B$. At each step, choose an arbitrary edge $xy \in B \setminus B'$. By the definition of B , there is exactly one edge $tz \in A$ such that exactly one of the xt , yt , xz or yz is an edge in G . With no loss of generality, assume that $xt \in E(G)$. Consider two following cases.

Case 1. If $g(tz) = 1$, then change $g(xy)$, $g(xt)$ and $g(tz)$ to 0, 2 and 0, respectively. By this modification, $w(g)$ decreases by one. Add xy to B' and each edge $x'y' \in B$ to B' if either $yx' \in E(G)$ or $zx' \in E(G)$. There are at most $2\Delta - 2$ edges such as $x'y' \in B$ and so we add $2\Delta - 1$ edges to B' . Note that the vertex t becomes 2-covered.

Case 2. If $g(tz) \neq 1$, then $g(tz) = 0$. In this case, change $g(xy)$ from 2 to 1 and add each edge $x'y' \in B$ to B' if $xx' \in E(G)$ or $yx' \in E(G)$ and also add xy itself. There are at most $2\Delta - 2$ edges such as $x'y' \in B$ and so we add $2\Delta - 1$ edges to B' and $w(g)$ is reduced by one in this case too.

Now, we prove that after each step, g remains an ERDF. In the beginning, for every edge $e \in B$, both end points of e are 2-covered. In each step, for every vertex $v \in V(\langle B \setminus B' \rangle)$ which is not already 2-covered, we added the edge incident with v in B to B' . Also, for every $v \in V(G)$ which is not already 2-covered, we added all edges $rs \in B$ such that $vr \in E(G)$ or $vs \in E(G)$ to B' . So all vertices of $\langle B \setminus B' \rangle$ remains 2-covered after each step.

In Case 1, after each step, the only edge in A for which its value has been changed is tz and tz remains covered since $g(xt) = 2$. Consider an edge pq such that $p \in V(\langle A \rangle)$ and $q \in V(\langle B \rangle)$. The only bad case is whenever $q = y$, because all vertices in $\langle B \rangle$ are 2-covered except y . This is a contradiction since by assumption, there is no edge incident with xy having one end in A . For each edge pq , $p, q \in V(\langle B \setminus B' \rangle)$, both p and q are 2-covered, so by changing $g(xy)$ to 0, pq remains covered. All edges yr , $r \in B'$ are covered too, because by the assumption if r is not 2-covered, then xy is in B' too, a contradiction.

In Case 2, after each step, if an edge pq becomes uncovered, then $p \in \{x, y\}$.

If $q \in A$, then $pq = xt$ (because there is at most one such edge). The vertex t is 2-covered and so pq is covered, because as we stated before, all adjacent vertices of a vertex which are not 2-covered are added to B' . If $q \notin A$, then q is 2-covered as stated before and so pq is covered.

The procedure continues at least $\frac{|B|}{2\Delta-1}$ steps, thus the claim is proved and $w(g)$ is reduced at least by $\frac{|B|}{2\Delta-1}$.

Let $a = |V(\langle A \rangle)|$, $b = |V(\langle B \rangle)|$ and $c = |V(\langle C \rangle)|$. We have $c = n - a - b$. There are at most $a(\Delta - 1)$ edges with exactly one end in A and there are at least $\frac{b}{2} + c$ edges with exactly one end point in B or C and the other end point in A . So we have:

$$a(\Delta - 1) \geq \frac{b}{2} + c$$

and hence

$$b \geq 2n - 2\Delta a.$$

Also, we find that $w(g) \leq \frac{a}{2} + b + c - \frac{b}{2(2\Delta-1)} = n - \frac{a}{2} - \frac{b}{2(2\Delta-1)}$. By substitution, we have:

$$w(g) \leq n - \frac{a}{2} - \frac{2n - 2\Delta a}{2(2\Delta - 1)} \leq \frac{2\Delta - 2}{2\Delta - 1}n + \frac{a}{2(2\Delta - 1)}.$$

By assumption, $a \leq \frac{n}{\Delta}$, so:

$$w(g) \leq \frac{2\Delta - 2}{2\Delta - 1}n + \frac{\frac{n}{\Delta}}{2(2\Delta - 1)} = \frac{2\Delta - 1}{2\Delta}n.$$

The proof is complete. □

Theorem 7. *For every graph G of order n and maximum degree Δ , $\gamma_{LR}(G) \leq \frac{2\Delta}{2\Delta+1}n$.*

Proof. Consider a maximal matching M of G . Let $Z = V(G) \setminus V(\langle M \rangle)$. Let g be an ERDF, such that $g(uv) = 2$ if $uv \in M$ and $g(uv) = 0$ otherwise. If $|Z| \geq \frac{n}{2\Delta+1}$, then we are done. Otherwise, assume that X is the induced subgraph of G such that $V(X) \subseteq V([M])$ and for every edge $uv \in M$, u and v are in $V(X)$ if and only if u and v are adjacent to no vertex in Z . Let Y be the induced subgraph of G with vertex set $V(Y) = V(\langle M \rangle) \setminus X$. For every edge $uv \in M$, if $u, v \in Y$, then at least one of u or v is adjacent to one of the vertices in Z . Let $x = |V(X)|$, $y = |V(Y)|$ and $z = |Z|$. By Theorem 6, one can find an ERDF g' for the induced subgraph X with weight at most $\frac{2\Delta-1}{2\Delta}x$. Let $g(uv) = g'(uv)$, for every edge $uv \in X$, $g(uv) = 2$ if $uv \in M \setminus X$ and $g(uv) = 0$, otherwise. There are at most Δz edges with exactly one end in Z and there are at least $\frac{y}{2}$ edges with one end in Y and one end in Z (because by definition, each edge of the perfect matching M in Y is adjacent to at least one vertex in Z), so $y \leq 2\Delta z$. Now, we have:

$$w(g) \leq y + \frac{2\Delta - 1}{2\Delta}x = \frac{2\Delta - 1}{2\Delta}n + \frac{y}{2\Delta} - \frac{2\Delta - 1}{2\Delta}z$$

$$\begin{aligned} &\leq \frac{2\Delta-1}{2\Delta}n + \frac{2\Delta z}{2\Delta} - \frac{2\Delta-1}{2\Delta}z \leq \frac{2\Delta-1}{2\Delta}n + \frac{z}{2\Delta} \\ &\leq \frac{2\Delta-1}{2\Delta}n + \frac{n}{2\Delta(2\Delta+1)} \leq \frac{2\Delta}{2\Delta+1}n, \end{aligned}$$

The proof is complete. \square

Theorem 8. *For any graph G , there exists a minimum ERDF $f = (E_0, E_1, E_2)$ such that*

- (a) $E_1 \cup E_2$ is a maximal matching in G .
- (b) $G[E_1]$ is a matching.
- (c) There is no edges between $V[E_1]$ and $V(G) \setminus (V[E_1] \cup V[E_2])$.

The minimum ERDF that satisfies the condition of the previous theorem is called *minimum standard ERDF*.

Theorem 9. *There is an infinite family of 3-regular graph G_n of order n such that*

$$\gamma_{LR}(G) = \frac{3n}{4}.$$

5 Edge Roman domination in trees

Theorem 10. *For a tree T of order n , the following hold*

$$\lceil \frac{2(n-l(T)+1)}{3} \rceil \leq \gamma_{LR}(T) \leq \lceil \frac{2(n-1)}{3} \rceil,$$

and the equality holds if and only if $T = P_n$.

Proof. For the upper bound, we apply induction on n . The base step handles trees with few vertices or small diameter. If $n = 1$, or 2, then the theorem is obvious. If $\text{diam}(T) = 2$ or 3, then T has a dominating edge, and $\gamma_{LR}(T) = 2 \leq \lceil 2(n-1)/3 \rceil$.

Hence we may assume that $\text{diam}(T) \geq 4$. For a subtree T' with n' vertices the induction hypothesis yields an ERDF f' with weight at most $\lceil \frac{2(n'-1)}{3} \rceil$. We find a suitable subtree T such that adding a bit more weight yields a small enough ERDF f for T .

Let P be a longest path in T chosen to maximize the degree of its next-to-last vertex v , and let u and w be the non-leaf and leaf neighbor of v , respectively.

Case 1. $d_T(v) > 2$. Obtain T' by deleting w . Define f on $E(T)$ by letting $f(x) = f'(x)$ except for $f(vw) = 0$. Note that f is an ERDF for T and that $w(f) = w(f') \leq \lceil 2(n-2)/3 \rceil \leq \lceil 2(n-1)/3 \rceil$.

Case 2. $d_T(v) = d_T(u) = 2$. Obtain T' by deleting u, v and w . Let z be other neighbor of u . Define f on $E(T)$ by letting $f(x) = f'(x)$, except for

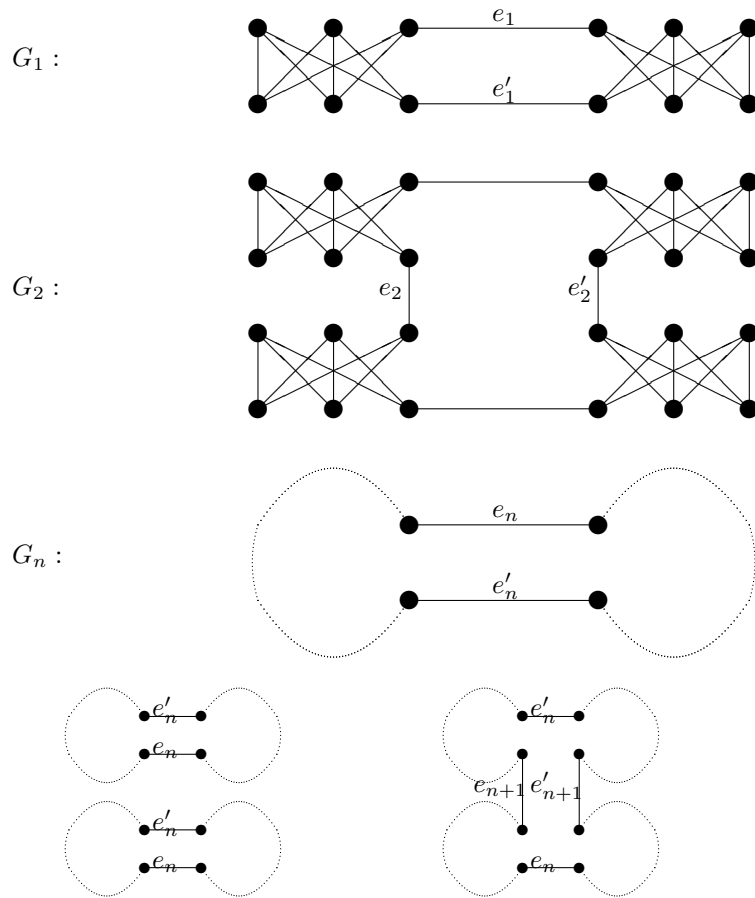


Figure 2 Proof of theorem 9

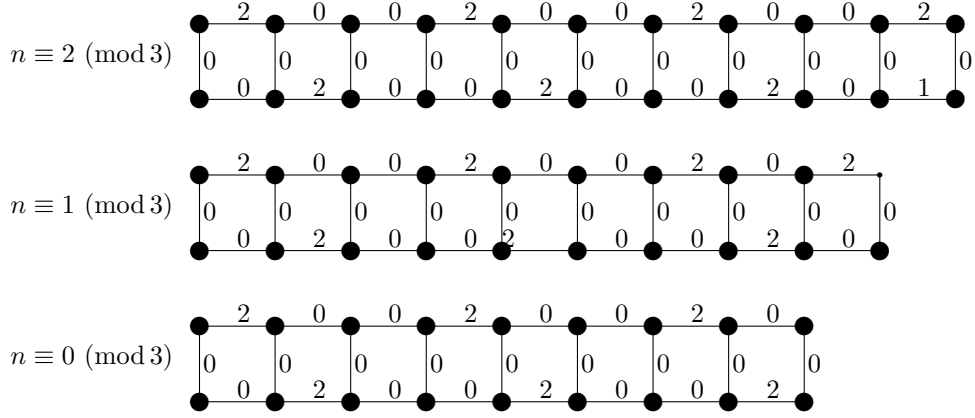


Figure 3

$f(zu) = f(vw) = 0$ and $f(uv) = 2$. Again f is an *ERDF* for T and that $w(f) = w(f') + 2 \leq \lceil 2(n-4)/3 \rceil + 2 = \lceil 2(n-1)/3 \rceil$.

Case 3. $d_T(u) > 2$ and every penultimate neighbor of u has degree 2. If every neighbor of u is penultimate or a leaf, then $\text{diam}(T) = 4$ and T is obtained from a star with center u by subdividing k edges, where $k \geq 2$. Put weight 2 on vu and weight 1 on the nonneighbor edges of u except vw . Now, $w(f) = k + 1$ and $n \geq 2k + 1 \geq 5$, so $w(f) \leq (n + 1)/2 \leq \lceil 2(n-1)/3 \rceil$. Otherwise, some neighbor t of u is neither penultimate nor a leaf. Obtain T' from T by deleting the vertices of the component of $T - tu$ containing u . Define f on $V(T)$ by $f(x) = f'(x)$, except for $f(uv) = 2$ and $f(x) = 1$ for each nonneighbor edges x of uv outside T' except vw . Again f is an *ERDF*. We have $w(f) = w(f') + k + 1$, where k is the number of leaves of T at distance 2 from u . $w(f) \leq \lceil 2(n-2k-2)/3 \rceil + k + 1 \leq \lceil 2(n-1)/3 \rceil$.

For the lower bound, we apply induction on n too. Similar to the first part of the proof, if $n = 1$, or 2 or $\text{diam}(T) = 2$ or 3 theorem is obvious. Let P, v, u and w be defined as before.

Case 1. $d_T(v) > 2$. Obtain T' by deleting w . T' is an induced subgraph of T and by Theorem ?? and induction hypothesis, $\gamma_{LR}(T) \geq \gamma_{LR}(T') \geq \lceil 2(n-1-l(T')+1)/3 \rceil = \lceil 2(n-l(T)+1)/3 \rceil$.

Case 2. $d_T(v) = 2$ and every penultimate neighbor of u has degree 2. Suppose that v has $k \geq 1$ penultimate and $s \geq 0$ leaf neighbor. If $\text{diam}(T) = 4$, then $\gamma_{LR}(T) = k + 1$. Otherwise, consider t and T' as before and let T_1, \dots, T_r be the component of $T' - t$. If f is a minimum standard *ERDF* of T , we can assume that $f(tu) = 0$ or $f(tu) = 2$. If $f(tu) = 0$, then the restriction of f to T' is a minimum *ERDF* f' for T' and by induction hypothesis $w(f) = k + 1 + w(f') \geq k + 1 + \lceil 2(n-k-s-l(T')+1)/3 \rceil \geq k + 1 + w(f') \geq k + 1 + \lceil 2(n-2k-s-1-(l(T)-k-s+1)+1)/3 \rceil \geq \lceil 2(n-l(T)+1)/3 \rceil$. If $f(tu) = 2$, then for $i, 1 \leq i \leq r$ the restriction of f to T_i is a minimum *ERDF*

f_i for T_i and similarly, $w(f) = k + 2 + \sum_{i=1}^r w(f_i) \geq \lceil 2(n - l(T) + 1)/3 \rceil$.

Note that the above discussion, the second part of theorem is easily obtained. \square

6 Edge Roman domination in some other classes of graphs

For a cycle $C = (E, V)$, where $V = \{v_1, \dots, v_n\}$ and $E = \{v_1v_2, \dots, v_{n-1}v_n\}$, a *standard ERDF*, $f = (E_0, E_1, E_2)$, is defined as follows:

- If $n \equiv 0 \pmod{3}$, $E_1 = \emptyset$, $E_2 = \{0 \leq i \leq (n-3)/3 | v_{3i+1}v_{3i+2}\}$ and $E_0 = E \setminus (E_1 \cup E_2)$.
- If $n \equiv 1 \pmod{3}$, $E_1 = \{v_{n-1}v_n\}$, $E_2 = \{0 \leq i \leq (n-4)/3 | v_{3i+1}v_{3i+2}\}$ and $E_0 = E \setminus (E_1 \cup E_2)$.
- If $n \equiv 2 \pmod{3}$, $E_1 = \emptyset$, $E_2 = \{0 \leq i \leq (n+1)/3 | v_{3i+1}v_{3i+2}\}$ and $E_0 = E \setminus (E_1 \cup E_2)$.

Example 1. A 3-regular graph of order $10n$ and Roman edge domination number $6n$. Consider a cycle of order $4n$ with vertices v_1, \dots, v_{4n} . Adding new vertices w_1, \dots, w_n and for each i , $1 \leq i \leq n$ join w_i to v_{4i-3}, v_{4i-1} to obtain H_n . Suppose that H'_n is a copy of H_n with the vertices v'_1, \dots, v'_{4n} and w'_1, \dots, w'_n . For each i , $1 \leq i \leq n$, join w_i to w'_i , and for every i , $1 \leq i \leq 2n$ join v'_{2i} to v'_{2i} . We obtain a graph G_n of order $5n$ which is decomposable to H_1 and by Theorem 5 $\gamma_{LR}(G) = \frac{3n}{5}$.

Lemma 11. [1] $\gamma_R(P_n) = \lceil \frac{2n}{3} \rceil$, $\gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$.

[7].

Lemma 12. [7] $\gamma_{LR}(P_n) = \lceil \frac{2(n-1)}{3} \rceil$, $\gamma_{LR}(C_n) = \lceil \frac{2n}{3} \rceil$.

Theorem 13. For $n \geq 2$, $\gamma_{LR}(P_2 \times P_n) = \lceil \frac{4n}{3} \rceil$.

Proof. Let $P_2 = (\{u_1, u_2\}, \{u_1u_2\})$, $P_n = (\{v_1, \dots, v_n\}, \{v_1v_2, \dots, v_{n-1}v_n\})$. Obviously, the function $f : E(P_2 \times P_n) \rightarrow \{0, 1, 2\}$ as follows

$$f(e) = \begin{cases} 2 & \text{if } e = \{u_1v_{3i+1}, u_1v_{3i+2}\} \text{ or } e = \{u_2v_{3i+2}, u_2v_{3i+3}\} \text{ or} \\ & \text{if } n \equiv 1 \pmod{3} \text{ and } e = \{u_2v_{n-1}, u_2v_n\}, \\ 1 & \text{if } n \equiv 2 \pmod{3} \text{ and } e = \{u_2v_{n-1}, u_2v_n\}, \\ 0 & \text{otherwise} \end{cases}$$

is a Roman edge function (see Figure 3). Therefore $\gamma_{LR}(P_2 \times P_n) \leq f(E(P_2 \times P_n)) = \lceil \frac{4n}{3} \rceil$. So by induction on n , we prove that $\gamma_{LR}(P_2 \times P_n) \geq \lceil \frac{4n}{3} \rceil$. The base step is obvious. Suppose that $f : E \rightarrow \{0, 1, 2\}$ is a γ_{LR} -function. If for one of $n-2$ middle step edges of $P_2 \times P_n$, e , $f(e) = 2$, then all four neighbor edges of e have label 0 and the restriction of f to $E \setminus N[e]$ is an ERDF for

the graph $G \setminus N[e]$. Clearly, $G \setminus N[e]$ is $(P_2 \times P_i) \cup (P_2 \times P_{n-1-i})$, for some i , $1 < i < n$. We have $\gamma_{LR}(P_2 \times P_n) \geq \gamma_{LR}((P_2 \times P_i) \cup (P_2 \times P_{n-1-i})) + 2$. For i , $2 < i < n - 2$, by the inductive hypothesis:

$$\begin{aligned} \gamma_{LR}(P_2 \times P_n) &\geq \gamma_{LR}((P_2 \times P_i) \cup (P_2 \times P_{n-1-i})) + 2 \\ &\geq \lceil \frac{4i}{3} \rceil + \lceil \frac{4(n-1-i)}{3} \rceil + 2 \\ &\geq \lceil \frac{4(n-1)}{3} \rceil + 2 \geq \lceil \frac{4n}{3} \rceil, \end{aligned}$$

and for $i = 2$ or $n - 1$ by the inductive hypothesis and Lemma 12 we have,

$$\begin{aligned} \gamma_{LR}(P_2 \times P_n) &\geq \gamma_{LR}((P_2 \times P_1) \cup (P_2 \times P_{n-2})) + 2 \\ &\geq 1 + \lceil \frac{4(n-2)}{3} \rceil + 2 \geq \lceil \frac{4n}{3} \rceil. \end{aligned}$$

Otherwise, every $n - 2$ middle step edges of $P_2 \times P_n$ has label 0 or 1. By the deleting of these $n - 2$ middle step edges from $P_2 \times P_n$, C_{2n} is obtained and the induced f on it is a Roman edge function. Therefore, $\gamma_{LR}(P_2 \times P_n) = \gamma_{LR}(C_{2n}) = \lceil \frac{4n}{3} \rceil$ and we are done. \square

Theorem 14. For $n \geq 1$, we have $\gamma_{LR}(P_3 \times P_n) = 2n$.

Proof. Let $P_3 = (\{u_1, u_2, u_3\}, \{u_1u_2, u_2u_3\})$, $P_n = (\{v_1, \dots, v_n\}, \{v_1v_2, \dots, v_{n-1}v_n\})$. Obviously, the function $f : E(P_3 \times P_n) \rightarrow \{0, 1, 2\}$ defined as

$$f(e) = \begin{cases} 2 & \text{if } e = \{u_2v_{3i+1}, u_2v_{3i+2}\}, e = \{u_1v_{3i+2}, u_1v_{3i+3}\} \text{ or } e = \{u_3v_{3i+2}, u_3v_{3i+3}\}, \\ 1 & \text{if } n \equiv 1 \pmod{3} \text{ and } e = \{u_1v_{n-1}, u_1v_n\} \text{ or } e = \{u_3v_{n-1}, u_3v_n\} \\ 1 & \text{if } n \equiv 2 \pmod{3} \text{ and } e = \{u_1v_{n-1}, u_1v_n\} \text{ or } e = \{u_3v_{n-1}, u_3v_n\}, \\ 0 & \text{otherwise} \end{cases}$$

is a Roman edge function. Therefore we find $\gamma_{LR}(P_3 \times P_n) \leq f(E(P_3 \times P_n)) = 2n$. By induction on n , we prove that $\gamma_{LR}(P_3 \times P_n) \geq 2n$. The base step is obvious. Suppose that $f : E(P_3 \times P_n) \rightarrow \{0, 1, 2\}$ is a γ_{LR} -function. If for some i , $1 \leq i \leq n$ the edge $e_i = \{u_1v_i, u_2v_i\}$, $f(e_i) = 2$, then all of its neighbor edges of e_i have label 0 and the restriction of f to $E(P_3 \times P_n) \setminus N[e_i]$ is an *ERDF* for the graph $(P_3 \times P_n) \setminus N[e_i]$. Clearly, $(P_3 \times P_i) \cup (P_3 \times P_{n-1-i})$, for some i , $1 \leq i \leq n$ is an induced subgraph of $G \setminus N[e]$. By Theorem ??, $\gamma_{LR}(P_3 \times P_3) \geq \gamma_{LR}((P_3 \times P_i) \cup (P_3 \times P_{n-1-i})) + 2$. By the induction hypothesis:

$$\begin{aligned} \gamma_{LR}(P_3 \times P_n) &\geq \gamma_{LR}((P_3 \times P_i) \cup (P_3 \times P_{n-1-i})) + 2 \\ &\geq 2i + 2(n - 1 - i) + 2 = 2n. \end{aligned}$$

Otherwise, for every i , $1 \leq i \leq n$ $f(e_i) = 0$ or 1. By deleting these n edges from $P_3 \times P_n$, $(P_2 \times P_n) \cup P_n$ is obtained and the induced f on it is a Roman edge function. Therefore, $\gamma_{LR}(P_3 \times P_n) \geq \gamma_{LR}((P_2 \times P_n) \cup P_n) = \lceil \frac{4n}{3} \rceil + \lceil \frac{2(n-1)}{3} \rceil = 2n$ and we are done. \square

Corollary 1. If $3 \mid mn$, then $\gamma_{LR}(P_m \times P_n) \leq \frac{2mn}{3}$.

Proof. Suppose that $3 \mid n$. Let $P_m = (\{u_1, \dots, u_m\}, \{u_1u_2, \dots, u_{m-1}u_m\})$, $P_n = (\{v_1, \dots, v_n\}, \{v_1v_2, \dots, v_{n-1}v_n\})$. Obviously the function $f : E(P_m \times P_n) \rightarrow \{0, 1, 2\}$ define as follows:

$$f(e) = \begin{cases} 2 & \text{if } e = \{u_jv_{3i+1}, u_jv_{3i+2}\}, \text{ and } 2 \mid j \\ & \text{or } e = \{u_jv_{3i+2}, u_jv_{3i+3}\}, \text{ and } 2 \nmid j \\ 0 & \text{otherwise} \end{cases}$$

is a Roman edge function. Therefore $\gamma_{LR}(P_m \times P_n) \leq f(E(P_m \times P_n)) = \frac{2mn}{3}$. \square

Theorem 15. *For every graph G ,*

$$m(G \times K_2 \times K_2) \leq 2m(G \times K_2).$$

Proof. Let G be a graph. Suppose that G_1 and G_2 are two copies of G such that G_1 has vertex set $V = \{v_1, \dots, v_n\}$ and G_2 on vertex set $U = \{u_1, \dots, u_n\}$, and $f : V \rightarrow U$, $f(v_i) = u_i$ ($1 \leq i \leq n$), is a graph isomorphism. For i , $1 \leq i \leq n$, join v_i to u_i to obtain $G \times K_2$. Now, let $(G \times K_2)^*$ be a copy of $G \times K_2$ with vertex set $U' \cup V'$ ($V' = \{v'_1, \dots, v'_n\}$ and $U' = \{u'_1, \dots, u'_n\}$) and $g : V \cup U \rightarrow V' \cup U'$, $f(v_i) = v'_i$ and $f(u_i) = u'_i$ ($1 \leq i \leq n$), is a graph isomorphism. For i , $1 \leq i \leq n$, join v_i to v'_i and u_i to u'_i to obtain $G \times K_2 \times K_2$.

Let M be a maximal matching of $G \times K_2$. We claim that $f(M) \cup g(M)$ is a maximal matching of $G \times K_2 \times K_2$. Obviously, $f(M) \cup g(M)$ is a matching. Suppose that $e \notin f(M) \cup g(M)$. If $e \in E(G \times K_2)$ (or $E((G \times K_2)^*)$), then e is adjacent to $f(M)$ (or $g(M)$). So, without loss of generality, assume that $e = \{u_i, u'_i\}$. If u_i is incidence to $f(M)$ or u'_i is incidence to $g(M)$ we are done. Otherwise, u_i is not incidence to $f(M)$ and u'_i is not incidence to $g(M)$. Hence, u_i is not incidence to M and v_i is not incidence to M and therefore, v_iu_i is not incidence to M in $G \times K_2$, a contradiction.

By definition

$$m(G \times K_2 \times K_2) \leq |f(M) \cup g(M)| = 2|M|,$$

and if M be a minimum maximal matching of $G \times K_2$, the theorem is proved. \square

Theorem 16. *For every graph G ,*

$$\gamma_{LR}(G \times K_2 \times K_2) \leq 2\gamma_{LR}(G \times K_2).$$

Proof. Consider the notations of the previous theorem. Let $h = (E_0, E_1, E_2)$ be an *ERDF* of $G \times K_2$. We claim that $h' = (\dots, f(E_1) \cup g(E_1), f(E_2) \cup g(E_2))$ is an *ERDF* for $G \times K_2 \times K_2$. If $e \in E(G \times K_2)$ (or $E((G \times K_2)^*)$) and $h'(e) = 0$, then $h(f^-(e)) = 0$ (or $h(g^-(e)) = 0$). So, $f^-(e)$ (or $g^-(e)$) is incidence to E_2 and therefore e is incidence to $f(E_2)$ (or $g(E_2)$). So, without loss of generality, $e = \{u_i, u'_i\}$. If u_i is incidence to $f(E_2)$ or u'_i is incidence to $g(E_2)$ we are done. Otherwise, u_i is not incidence to $f(E_2)$ and u'_i is not incidence to $g(E_2)$. Hence,

u_i is not incidence to E_2 and v_i is not incidence to E_2 and therefore, $v_i u_i$ is not incidence to E_2 in $G \times K_2$, a contradiction.

By definition

$$\gamma_{LR}(G \times K_2 \times K_2) \leq 2|f(E_2) \cup g(E_2)| + |f(E_1) \cup g(E_1)| = 4|E_2| + 2|E_1|,$$

and if h be a minimum *ERDF* of $G \times K_2$, the theorem is proved. \square

Theorem 17. *If $h = (E_0, E_1, E_2)$ is an *ERDF* for Q_n such that $E_1 \cup E_2$ is a maximal matching for Q_n , then at least one third of the remaining edges have the property of being adjacent to two edges of $E_1 \cup E_2$.*

Theorem 18. *For each $n \geq 1$, the following holds:*

$$\gamma_{LR}(Q_n) \geq \frac{2^{n+1}n}{3n-1}.$$

Proof. Consider the notations of the previous Theorem. For $n = 1, \text{ or } 2$, it is obvious. Suppose that $n \geq 3$. Let m_1 (or m_2) be the number of edges of Q_n with label 0 such that is adjacent to two (just one) edges of $E_1 \cup E_2$. By previous theorem,

$$0 \leq 2m_1 - m_2 = 3(2m_1 + m_2) - 4(m_1 + m_2)$$

We have

$$m_1 + m_2 = 2^{n-1}n - |E_1| - |E_2| - 2(n-1)|E_1|.$$

Also, since every vertex of E_2 have degree n

$$2m_1 + m_2 = 2n - 1|E_2| - 2(n-1)|E_1|.$$

It is not hard to see

$$\begin{aligned} 0 &\leq 2(3n-1)|E_2| + (2n+2)|E_1| - 2^{n+1}n \\ &\leq 2(3n-1)|E_2| + (3n-1)|E_1| - 2^{n+1}n, \end{aligned}$$

and this prove the theorem. \square

7 Computational results

It seems that $\gamma_{RL}(G)$ is never greater than $\frac{\Delta}{\Delta+1}n$ in all graphs. So we propose the following conjecture:

Conjecture. *For a graph G of order n and maximum degree Δ , $\gamma_{RL}(G) \leq \lceil \frac{\Delta}{\Delta+1}n \rceil$.*

Table 1. All connected graphs

n	No. of Graphs	Max(Min(E_1))	No. of graphs for which		
			$\gamma_{LR} = \lfloor \frac{\Delta}{\Delta+1}n \rfloor$	$\gamma_{LR} = \lceil \frac{\Delta}{\Delta+1}n \rceil$	$\gamma_{LR} > \lceil \frac{\Delta}{\Delta+1}n \rceil$
2	1	1	1	0	0
3	2	0	1	1	0
4	6	1	3	1	0
5	21	1	9	3	0
6	112	1	57	1	0
7	853	2	137	5	0
8	11117	2	1528	1	0
9	261080	3	1016	3	0
10	11716571	3	18840	4	0

Table 2. 3-regular connected graphs

n	No. of Graphs	Max(Min(E_1))	No. of graphs for which		
			$\gamma_{LR} = \lfloor \frac{3n}{4} \rfloor$	$\gamma_{LR} = \lceil \frac{3n}{4} \rceil$	$\gamma_{LR} > \lceil \frac{3n}{4} \rceil$
4	1	1	1	0	0
6	2	1	1	1	0
8	5	0	5	0	0
10	19	1	9	1	0
12	85	1	6	0	0
14	509	1	220	0	0
16	4060	1	42	0	0
18	41301	2	1045	4	0
20	510489	?	?	?	?

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