# Lying Property in Balance Theory

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**Abstract.** Structural balance theory attempts to model signed networks consisting of positive (friendly) and negative (antagonistic) relations. We consider network creation games for modeling the evolution of signed networks. Recently, Malekzadeh et. al. (KDD 2011) have proposed a network creation game based on structural balance theory of signed networks. In this paper, we introduce a generalized model based on the prize collecting framework in which people can lie about their relations while all other properties of the previous model are retained.
First, we prove that playing best response is NP-hard. Then, we characterize many structural properties of Nash equilibria networks of the game such as upper bound of 2 for diameter, an upper bound of $|V(G)|/2$ for the size of a maximum independent set and lower bound of $|V(G)|/2$, where $G$ is the underlying Nash equilibrium graph. By constructing examples, we show that all of our bounds are asymptotically tight. Then, we discuss the convergence issues of the game and show that the game with any initial state will eventually converge to a Nash equilibrium in a polynomial number of rounds. In the final part of this paper, we define a new parameter Price of Lying (PoL) which shows the effects of honesty in social welfare. By experimenting our model on social networks' real data, we show that a society without lies will be happier even if its members are fully selfish.

## 1 Introduction

The majority of the research in the field of social network analysis has centered on models consisting only one type of relations among actors, but in recent works, two types of relations are considered: *Positive* and *Negative* [15, 24]. Positive relations depict positive aspects such as friendship, like, trust or follow, while a negative relation shows antagonism, distrust or unlike. To model social networks with positive and negative relations, we use *Signed Graphs* in which every edge admits either a positive or a negative sign.

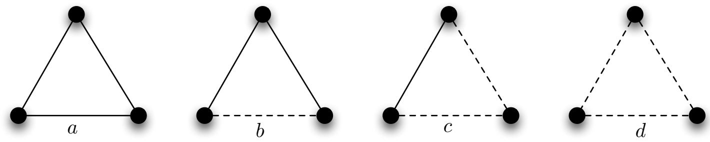


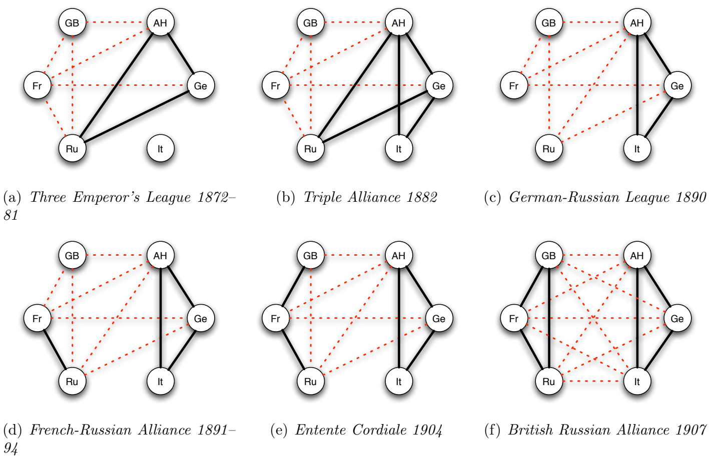

**Fig. 1.** Possible combinations of edge signs of an undirected triangle: triangles (a) and (c) are balanced, but triangles (b) and (d) are unbalanced. Full and dashed lines represent positive and negative relations respectively [25].

*Balance Theory*, first introduced by Heider [17], claims that some combination of local relationships among actors are more stable than other combinations and the actors of a social group have an unconscious tendency to change their relationships to reach more stable states. Harary formalized this theory [16] and in a joint work with Cartwright, proposed *Structural Balance Theory (SBT)* which considers triadic relations among people and classifies them into two classes based on BT: *balanced* and *unbalanced* [8]. A balanced triangle shows relations between three people without mental disturbance, such as Figure 1-a that shows three people who are mutually friends and Figure 1-c that shows two friends with a common enemy. In an unbalanced triangle, we have a psychological stress on players involved. In Figure 1-b, we see a pair of enemies with a common friend. In this situation, the common friend tries to force enemies to change their relationship to a friendship; or else, one of the enemies tries to force the common friend to break his friendship with the other one. Figure 1-d shows another unbalanced state in which three people are all enemies and there is a chance that two of them join together against the third person.

Davis proposes a new variant of SBT called *Weak Balance Theory (WBT)* in which, triangles of type 1-d (with 3 negative edges) are excluded from unbalanced class [10]. This setting was based on the fact that "the enemy of my enemy is a friend of mine" is not necessarily true. Leskovec et al. show that the number of triangles with three negative links in online social media are much more than those in randomly generated graphs [24]. These results confirms the naturality and the rationality of WBT proposed by Davis in the online social media.

Studying the evolution of signed networks has many applications in social sciences such as political studies of international relations. Research in political science has shown that SBT can provide an effective explanation of the behavior of nations during various international crises. For example, Antal et al. use the shifting alliances preceding World War I as an example of the evolution process of a signed network based on SBT [2] – see Figure 2.

*Network creation games* have been widely used in order to characterize how network structures evolve and how they are affected by individuals' decisions when network's nodes are selfish agents [19]. There are several network creation games which take different networks' parameters into account [1, 3, 6, 7, 9, 11–14, 18, 20, 27, 29]. Arount van de Rijt attempts to investigate the structure of signed graphs using SBT [32]. He proposes a best-response model with utility function of each node equal to number of balanced triangles containing the node. He proves theorems about balanced and stochastically stable graphs in the model. The




(a) *Three Emperor's League 1872–81* (b) *Triple Alliance 1882* (c) *German-Russian League 1890*

(d) *French-Russian Alliance 1891–94* (e) *Entente Cordiale 1904* (f) *British Russian Alliance 1907*

**Fig. 2.** The evolution of alliances in Europe, 1872-1907 (the nations GB, Fr, Ru, It, Ge, and AH are Great Britain, France, Russia, Italy, Germany, and Austria-Hungary respectively). Solid dark edges indicate friendship and dotted red edges indicate enemity. Note how the network slides into a balanced labeling i.e. into the World War I. This figure and example are from Antal, Krapivsky, and Redner [2].

main weakness of this work is studying the evolution of sign changes while leaving the network structure fixed.

In [25], authors propose a network creation game based on SBT. In their model, for each $1 \leq i \leq |V(G)|$, utility of the $i$th node is defined to be $(\triangle_{b_i} - \bigtriangledown_{ub_i})$ where $\triangle_{b_i}$ and $\bigtriangledown_{ub_i}$ are the number of balanced and unbalanced triangles containing the $i$'th node respectively. The strategy of each player is to choose a subset of vertices and create signed links to them so as to maximize its utility function. In this model, the NP-hardness of finding the best-response is proved and it is shown that it will converge in $O(|V(G)|)$ number of rounds to a Nash equilibrium graph $G$ which is always a complete graph. We call this game the *BT-Game*.

### 1.1 Our Contribution

In many real world situations, actors such as companies, political parties or countries deceive their competitors by lying about state of their relations to others as a part of their success policy, such as the secret protocol of the Molotov-Ribbentrop Pact during World War II, allotting Germany a larger part of Poland while ceding most of Lithuania to the Soviet Union [33] or many hidden contracts between companies to seize the market place. This lying phenomena can be modeled also in BT-Game by using prize collecting framework which has been studied in many contexts specially in network design [4, 5, 26, 28]. In this paper, we study a generalized version of BT-Game which retains all the properties and more than that, it can formulate how hidden relations form in a social network. We call this generalized version of BT, *Prize Collecting Balance Theory (PCBT)* and so we call the network creation game based on this theory the *PCBT-Game*.

More precisely, in PCBT-Game, an actor collaborates with others by making relations with other actors and deciding whether he wants to lie about his relations or not. We assume that the game is played in rounds and each player makes its decisions independently in its own turn. The strategy of making decision for state of a relation is the same for both of its end points, i.e. if one side of a relation decides to change the relation's sign or lie to other actors about it, then the other side of the relation does the same. Thus, lying about a relation to other actors in the network incurs a cost to both ends of the relation and the actors involved in that relation cannot lie to each other or themselves about its state. In section 4 of this paper, we prove that PCBT-Game always converges to a stable state where no one can increase its utility, so we can conclude that the actors' ability to force their own strategy for a relation to the other side of that relation does not reduce the game's naturality.

In this paper, we first show that finding the best-response of players in PCBT-Game is NP-hard (Theorem 1). Then we consider the structural properties of the Nash equilibrium graphs and obtain the following results. For every Nash equilibrium graph $G$, we have:

- small-world property: The diameter of $G$ is less than or equal to 2 (Theorem 2).
- maximum independent set size: The size of maximum independent set of $G$ is at most $\frac{|V(G)|}{2}$ (Theorem 3).
- minimum number of edges: The number of edges in $G$ is at least $\frac{|V(G)|^2}{4}$ (Corollary 3).
- minimum degree: Minimum degree of $G$ is at least $\frac{|V(G)|}{2}$ (Corollary 2).

By providing an example, we show that all these properties are almost tight (Theorem 4). Then, we show that convergence time of PCBT-Game is polynomially bounded (Theorem 5). In the final part of this paper, we define a new measure, Price of Lying (PoL), which shows the effects of honesty in social welfare. By experimenting our model on social networks' real data, we show that a society without lies would be happier even when its members are fully selfish.

## 2 Model and Preliminaries

We consider a network creation game with $n$ players labeled by $1, \ldots, n$ which are vertices of a signed graph $G$. The graph $G$ is called the *Game-graph.* The set vertices and edges of the graph are denoted by $V(G)$ and $E(G)$ respectively. $N(v)$ is the set of neighbors of a vertex $v$ and $d(v) = |N(v)|$. Define $v$-triangle as a triangle having $v$ as one of its vertices and $uv$-triangle a triangle having both $u$ and $v$ as two of its vertices.

### 2.1 PCBT-Game

Player $i$ $(1 \leq i \leq n)$ plays his best response at times $t = kn + i$ for $k \geq 0$ until the game converges to a Nash equilibrium i.e. when all players are playing their best responses assuming that strategy of other players are fixed. Game starts with an initial setting for $G$ in which each vertex has at least one adjacent signed edge. Throughout this paper, we show the sign of the edge between vertices $u, v \in V(G)$ by function $\texttt{link} : V(G) \times V(G) \rightarrow \{-1, 0, +1\}$. More precisely, $\texttt{link}(u, v)$ shows the sign of edge $uv$ where its value is either $+1$ demonstrating positive sign, $-1$ demonstrating negative sign or 0 demonstrating no edge exists between $u$ and $v$.

In each round, a node manipulates his adjacent edges i.e. he removes or adds new signed edges to other nodes or changes their signs. We assume that no vertex creates edges to itself. Also, it can lie about the signs of his adjacent edges to other nodes. Of course, no vertex lies about its adjacent edges to itself. Sign of each edge (values of $\texttt{link}$) may change during the game according to strategies of its end point vertices. So, strategy of a vertex $v \in V(G)$ is to specify $\texttt{link}(v, u)$ for all $u \in V(G)$ and whether he lies about each edge $uv$ to each node $z \in V(G)$ or not. The function $\texttt{hide}_u : V(G) \times V(G) \rightarrow \{0, 1\}$ shows the lying strategy of vertices. If $u$ has lied about its edge which it shares with $v$ to $z$ then $\texttt{hide}_u(v, z) = 1$, otherwise $\texttt{hide}_u(v, z) = 0$. Note that when a vertex decides to lie about the edge $uv$ to $z$ in its playing round, then vertex $v$ must lie about $uv$ to $z$ too. The values of function $\texttt{link}$ is also symmetric i.e. $\texttt{link}(u, v) = \texttt{link}(v, u)$ for all $u, v \in V(G)$.

Lying about an edge $uv$ to $z$ by either $u$ or $v$, incurs cost $C_u(v)$ to $u$ and $v$ regardless of which one of $u$ and $v$ has decided to lie about the edge. Note that $C_u(v) = C_v(u)$ for all $u, v \in V(G)$. For ease of presentation, we define the function $\mathtt{sign}_v : V(G) \times V(G) \rightarrow \{-1, 0, 1\}$. For each vertex $u \in V(G)$ and each edge $vz$, $\mathtt{sign}_u(v, z)$ shows sign of the edge $vz$ in view of node $u$. So, we have:

$$\mathtt{sign}_u(v,z) = \begin{cases} \mathtt{link}(v,z) & \text{if } \mathtt{hide}_v(z,u) = 0 \\ -\mathtt{link}(v,z) & \text{if } \mathtt{hide}_v(z,u) = 1 \end{cases}$$

In our model, for every vertex $u \in V(G)$, the amount of $u$'s happiness (utility, net worth) denoted by $NW(u)$ is defined as the number of balanced $u$-triangles minus the number of unbalanced $u$-triangles and sum of the costs that he pays to lie about its adjacent edges. Formally,

$$\begin{aligned} NW(u) = & \sum_{v,z \in V(G), u \neq v \neq z} \mathtt{sign}_z(u,v)\mathtt{sign}_v(u,z)\mathtt{sign}_u(v,z) \\ & - \sum_{v,z \in V(G), u \neq v \neq z} \mathtt{hide}_u(v,z) C_u(v). \end{aligned}$$

The first component of the r.h.s of the above formula counts the number of balanced u-triangles minus the number of unbalanced ones. We assume that each balanced triangle increases $u$'s happiness one unit and each unbalanced triangle costs him one unit. The second component is the cost of relations (edges) that he lied about them. The maximization problem defined here is called a *Net Worth Maximization* problem in the literature [21].

## 2.2 Preliminaries

Finding the best response in BT-Game is NP-hard [25]. It is easy to verify that by setting $C_u(v)$ a value more than 2 for all $u, v \in V(G)$, none of the vertices lie about any of their adjacent edges. So, finding the best-response of players in PCBT-Game is harder than BT-Game.

**Theorem 1.** *Finding the best response of a node in PCBT-Game is NP-hard.*

The following lemma shows relatively easy but yet important properties about the structure of $G$ after each round. $G[S]$ is the induced subgraph of $G$ on $S \subseteq V(G)$.

**Lemma 1.** *Assume that $u$ plays his best response and immediately after that we set $S = V(G) \setminus (N(u) \cup \{u\})$ where $G$ is the game-graph. Then we have*

a. *For any $v \in S$, the number of balanced and unbalanced triangles (in view of $u$) in graph $G$ created by adding the edge $uv$ with any sign are equal.*
b. $\forall v \in S, \forall z \in (V(G) \setminus \{u, v\}) : vz \in E(G) \Rightarrow uz \in E(G)$.
c. $E(G[S]) = \emptyset$.

*Proof.* a. W.l.o.g., assume that by adding $uv$ with positive sign, the value of $NW(u)$ will change and the number of new balanced and unbalanced triangles are equal to $x$ and $y$ respectively ($x, y \in \mathbb{N}$). We know that $u$ has played his best response, so by adding this edge, $NW(u)$ will not increase. Assume that $x \neq y$, so $x - y < 0$. Now, we add this edge with negative sign, every $uv$-triangle changes from balanced to unbalanced and vice versa, so the number of new balanced triangles will be more that the number of new unbalanced triangles and $NW(u)$ increases, a contradiction and the proof is complete.

b. Assume that $\exists v \in S, \exists z \in (V(G) \setminus \{u, v\}) : vz \in E(G) \wedge uz \notin E(G)$, so $u$ has no edges to $v$ or $z$ and there is a signed edge $vz$. We know from previous part that the number of $uz$-triangles is even ($2x$), and the number of $vz$-triangles is also even ($2y$). With the same approach it is easy to see that adding both $uz$ and $vz$ edges creates even number of $u$-triangles ($2w$). There is only one triangle that involves $uv$, $uz$ and $vz$, so $2x + 2y - 1 = 2w$. This is contrary to our assumption and the proof is complete.

c. By previous part, we know that $\forall v, z \in S : vz \notin E(G)$. So it is trivial that $G[S]$ is an independent set. □

## 3 Nash Equilibria and Properties

The validity of a network creation game arises from how much its Nash equilibria can satisfy general structural properties of social networks. So, the main focus of research in this area is to study properties of the Nash equilibrium graphs, specially the small-world property which is low diameter and high clustering-coefficient [1, 6, 7, 13, 14]. In this section, we study Nash equilibria of PCBT-Game and derive some of its most important structural properties. In BT-Game, it is shown that the Nash equilibria are complete graphs [25], but in PCBT-Game, we will show that this is not necessarily true. However, we show that under specific conditions for cost function, Nash equilibrium graph of PCBT-Game will be complete.

**Lemma 2.** *In a Nash equilibrium graph $G$, if $d(u) < |V(G)| - 1$ for a vertex $u \in V(G)$, then for every $v \in V(G) \setminus \{u\}$, $C_u(v) \geq 2$.*

*Proof.* Assume that there is a vertex $v \in V(G)$ such that $uv \notin E(G)$ and $\exists z \in V(G) : C_u(z) < 2$. First, we prove that $uz \in E(G)$. If $uz \notin E(G)$ then by Lemma 1 we know that by adding $uz$, $u$ will get equal number of balanced and unbalanced triangles (assume that this number is $x$). By adding $uz$ and lying about it to each vertex $p \in V(G)$ for which $up, zp \in E(G)$, we have:

$$\Delta NW(u) = x + (x - xC_u(z)) = 2x - xC_u(z) > 2x - 2x = 0$$

Where $\Delta NW(u)$ is the difference of the utility of $u$ after and before these modifications. Therefore, $NW(u)$ will increase. With the same approach it is easy to show that $vz \in E(G)$.

By Lemma 1 we know that by adding $uv$, number of new balanced and unbalanced $uv$-triangles will be equal. Now, $u$ can choose a sign for $uv$ such that $uvz$ triangle be unbalanced, by lying about $uz$ to $v$, $NW(u)$ changes to $NW(u) - (-1) + (1 - C_u(z))$ and we know that $C_u(z) < 2$ so $\Delta NW(u) > 0$, a contradiction and the proof is complete. □

The next corollary follows directly from the previous lemma.

**Corollary 1.** *In a Nash equilibrium graph $G$, if for all $u \in V(G)$, there exists a vertex $v \in V(G)$ $(u \neq v)$ such that $C_u(v) < 2$ then $G$ is either an empty or a complete graph.*

The next theorem shows an interesting property of every Nash equilibrium graph drawn from Lemma 1, which states that their diameter is $O(1)$.

**Theorem 2.** *Diameter of every Nash equilibrium that is not an empty graph, is at most 2.*

*Proof.* Assume that $u, v \in V(G)$, $dis(u, v) > 2$ and $d(u) + d(v) \neq 0$ ($dis(u, v)$ is the distance between $u$ and $v$). W.l.o.g., we assume that $d(v) \neq 0$. Hence $dis(u, v) > 2$ we have

$$\exists z \in V(G) : vz \in E(G) \wedge uz \notin E(G)$$

By Lemma 1 we know that if $uz \notin E(G)$ and $vz \notin E(G)$ then $uv \notin E(G)$, a contradiction and the proof is complete. □

The next theorem shows that there is not any independent set of size larger than $|V(G)|/2$ in any Nash equilibria. This implies that stable states of the game are almost dense graphs.

**Theorem 3.** *Size of a maximum independent set in every equilibrium $G$ is at most $\frac{|V(G)|}{2}$.*

*Proof.* Let $S$ be a maximum independent set of $G$ with $k$ vertices. First, we claim that for every $u \in S$ and $v \in V(G) \backslash S$, $uv \in E(G)$. Assume that $uv \notin E(G)$ then consider following two cases.

- If none of $v$'s neighbors are in $S$, then $S \cup \{v\}$ is an independent set with cardinality greater that $S$,which is a contradiction.
- If for some $z \in S$, $vz \in E(G)$, then $v, z \in V(G) \setminus N(u)$ and by Lemma 1, $v$ and $z$ can not be adjacent, which is also a contradiction.

Now, let $S = \{u_1, \ldots, u_k\}$, $V(G) \setminus S = \{v_1, \ldots, v_{n-k}\}$ and $|V(G)| = n$. For each vertex $u_i \in S$, we assign an $(n-k)$-dimensional vector $s_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,n-k})$ to $u_i$ such that for all $1 \leq j \leq n-k$, if $\texttt{link}(u, v)$ is positive then $s_{i,j} = 1$, otherwise, $s_{i,j} = -1$. Note that $\forall u, z_1, z_2 \in S, \forall v \in V(G) \setminus S : \texttt{sign}_{z_1}(u, v) = \texttt{sign}_{z_2}(u, v)$.

By Lemma 1, we know that adding the edge $u_i u_j$ with any sign for every $u_i, u_j \in S$, will not change the utility of $u_i$ and $v_j$. So, $s_i$ and $s_j$ has exactly $\frac{n-k}{2}$ equal elements and exactly $\frac{n-k}{2}$ unequal elements. The set of vectors $s_1, \ldots, s_k$ form rows of a Hadamard matrix i.e. every two different vectors represent two perpendicular vectors. This means that number of vectors are not greater than their dimension, so $k \leq n-k$, thus $k \leq \frac{n}{2}$ and the proof is complete. □

By Lemma 1, we know that for every vertex, the set of its non-adjacent vertices is an independent set. Using this fact and Theorem 3, we can obtain an $\Omega(n)$ lower bound on degree of all vertices in a Nash equilibrium graph of size $n$.

**Corollary 2.** *In a Nash equilibrium graph $G$, for every vertex $u \in V(G)$, we have $d(u) \geq \frac{|V(G)|}{2}$.*

Simply, an asymptotical lower bound of $\Omega(n^2)$ for the number of edges in a Nash equilibrium graph of order $n$ can be concluded.

**Corollary 3.** *For every Nash equilibrium graph $G$ we have $|E(G)| \geq \frac{|V(G)|^2}{4}$.*

In the next theorem, we show that for infinite number of $n$, there is a Nash equilibria of order $n$ with diameter 2 and an independent set of size $\frac{n}{3}$. So, set of all Nash equilibria includes a family of complex Nash equilibria of order $n$ which the size of their maximum independent set is $\Theta(n)$ and these Nash equilibria are not trivial such as complete graphs. Also, the next theorem shows that the upper bound of 2 for diameter of Nash equilibria provided by Theorem 2 is tight.

**Theorem 4.** *For infinitely many $n \in \mathbb{N}$, there is a Nash equilibrium graph with $n$ vertices and diameter exactly 2 and an independent set of size $\frac{n}{3}$.*

*Proof.* It is known that there exists infinite number of Hadamard matrices [30]. Now, for $n = 3p$ $(n, p \in \mathbb{N})$ consider a matrix $[m_{i,j}]_{\frac{n}{3} \times \frac{2n}{3}}$ such that its rows are any $\frac{n}{3}$ rows of a Hadamard matrix of $\frac{2n}{3} \times \frac{n}{3}$ (i.e. they are mutually orthogonal).

$$S = \{u_1, \ldots, u_p\}$$
$$T = \{v_1, \ldots, v_{n-p}\}$$
$$V(G) = S \cup T$$

and a cost function $C$ in which

$$\forall x \in S, \forall y \in V(G) : C(x, y) > 2$$
$$\forall x \in T, \forall y \in V(G) : C(x, y) = 0$$

Now, set signs of edges as follow

$$\forall u_i, u_j \in S, \forall z \in V(G) : \texttt{sign}_z(u_i, u_j) = 0$$
$$\forall u_i \in S, \forall v_j \in T, \forall z \in V(G) : \texttt{sign}_z(u_i, v_j) = m_{i,j}$$
$$\forall v_i, v_j, v_k \in T : \texttt{sign}_{v_k}(v_i, v_j) = 1$$
$$\forall v_i, v_j \in T, \forall u_k \in S : \texttt{sign}_{u_k}(v_i, v_j) = \texttt{sign}(u_k, v_i)\texttt{sign}(u_k, v_j)$$

We prove that with this setting, $G$ is a Nash equilibrium. Every vertex $v_i \in T$ has its maximum possible utility, since $v_i$ is connected to all other vertices and all of its triangles are balanced. Thus, we only need to prove that every vertex $u_i \in S$ is playing its best response. Consider a vertex $u_i \in S$, $u_i$ has $\frac{|T|(|T|-1)}{2}$ triangles and all of them are balanced. Now, assume that there exists a better response, say $\texttt{sign}'$, for $u_i$. Note that $u_i$ never lies about any edge since its cost function is always greater than 2. In $u_i$'s new strategy, assume that $u_i$'s edges remained unchanged to each vertex in $R \subseteq T$ in comparison to the $\texttt{sign}$ strategy and its edges to vertices $R' \subseteq T$ are changed. In this new strategy, at most $|R||S|$ number of $u_i$-triangles (containing one vertex from $T$) has changed state from unbalanced to balanced and at least $|R|(|T|-|R|)$ triangles (containing two vertices from $T$) has changed their state from balanced to unbalanced. Thus, if $\texttt{sign}'$ be a better strategy then $|R||S| > |R|(|T|-|R|)$, and so $|S| > |T|-|R|$. Now consider another strategy for all vertices, say $\texttt{sign}''$, with same edge signs as $\texttt{sign}$ strategy, except for vertex $u_i$ which has reversed values of edge signs e.i. $\texttt{sign}''_z(u_i, v_j) = -\texttt{sign}_z(u_i, v_j)$ for all $v_j \in T$ and $z \in V(G)$. It is easy to show that utility of all vertices in $\texttt{sign}''$ strategy is same as $\texttt{sign}$ strategy. So changing the $u_i$'s strategy to $\texttt{sign}'$ from $\texttt{sign}''$ will increase its utility. By similar justification, we have $|S| > |R|$. Thus,

$$2|S| > |T| \Rightarrow |S| > \frac{n}{3}$$

a contradiction, and the proof is complete. □

## 4 Convergence

Convergence is one of the most important issues about repeated games. It has been considered in many network creation games [22, 23, 25]. In [25], it is shown that BT-Game always converges to a complete graph in linear number of steps. In this section, we show that with every initial values of signs and costs, PCBT-Game converges to a Nash equilibrium in polynomial number of rounds.

**Theorem 5.** *If the minimum difference between possible utility values for all vertices $\epsilon$ is $\Omega(1)$, then PCBT-Game converges to a Nash equilibrium in at most $O(|V(G)|^4)$ rounds.*

*Proof.* Define the potential function of graph $\Phi$ as follow

$$\begin{aligned}\Phi(G) = \sum_{u,v,z \in V(G), u\neq v\neq z} & \big\{\texttt{sign}_z(u,v)\texttt{sign}_v(u,z)\texttt{sign}_u(v,z) \\ & + 1.5\big(\texttt{hide}_u(v,z)C_u(v) \\ & + \texttt{hide}_u(z,v)C_u(z)\big)\big\}\end{aligned}$$

We prove that if each vertex $u \in V(G)$ increases its utility by $\Delta$ after playing its best response, then the potential function will increase exactly by $3\Delta$. Consider modifications of $u$ in its strategy as a sequence of discrete actions on each edge and triangle.

First, we study states of triangles that $u$ is involved in them. If state of the triangle $uvz$ changes from unbalanced to balanced after $u$ performed a single action, then $NW(u)$ increases by 2, accordingly, for each one of $u$, $v$ and $z$, $\Phi(G)$ increases by 2 and totally increases by 6. Similarly, if state of the triangle $uvz$ changes from balanced to unbalanced, then $NW(u)$ and $\Phi(G)$ decrease by 2 and 6 respectively. If $u$ decides to change sign of $uv$ or $uz$ to zero then if $uvz$ was balanced, $NW(u)$ decreases by 1 and $\Phi(G)$ decreases by 3, otherwise, if $uvz$ was unbalanced, $NW(u)$ and $\Phi(G)$ increase by 1 and 3 respectively.

Now, we study states of $u$'s adjacent edges. If $u$ lies about $uv$ to $z$ in a single action, then $NW(u)$ will decrease by $C_u(v)$, we know that $u$ forces $v$ to lie about $uv$ to $z$ too. So, $\Phi(G)$ will decrease by $1.5C_u(v) + 1.5C_v(u) = 3C_u(v)$. Similarly, if $u$ tells the truth about $uv$ to $z$, $NW(u)$ and $\Phi(G)$ will increase by $C_u(v)$ and $3C_u(v)$ respectively. Note that in case $uv$ has been deleted by $u$ in previous actions, W.l.o.g., we assume that $u$ tells the truth about $uv$ before deleting it.

Therefore, if $\Delta > 0$ then $\Phi(G)$ will increase. Assume that $|V(G)| = n$, utility of each vertex is at most $\binom{n-1}{2}$, so the utility of each vertex will increase at most $\frac{1}{\epsilon}\binom{n-1}{2}$ times. In $n$ rounds, at least one player will increase its utility, otherwise, the game has converged to a Nash equilibrium. So, the game will go on for at most $\frac{n^2(n-1)(n-2)}{2\epsilon} = O(n^4)$ rounds and the proof is complete. □

# 5 Experimental Results

In this section we present results from computational experiments on randomly generated networks as well as real world social networks' datasets and we give interpretation for these results and the interesting observations which follow them. These experiments are focused on the effects of players lying and its occurrence frequency on the dynamics of the PCBT network creation games. For this purpose, we consider two simulation scenarios. In the first scenario we apply datasets from social networks, namely Epinions and Slashdot. The reason for choosing these social networks is that they have particular notions of negative and positive relations among their users, therefore, having a signed network structure. Note that in this context the notions, graph, node and edge are used interchangeably with network, player or user and relation, respectively. In the second scenario, we carry out games with different parameters (e.g. costs for hiding relationships among players) on diverse generated initial networks. The outcomes of these games demonstrate an interesting correspondence between the overall pleasures of players and the overall costs to hide these relations.

In the previous sections it was stated and proved that computing the best response in PCBT-game is NP-hard. Thus, in the simulation scenarios, instead of finding the optimum strategy for each player we look for an strategy proximate to optimal one by applying a polynomial time hill climbing local search approach. Using different heuristics for the local search scheme and comparing the yielded results to the computed best responses in the case that the number of nodes in the networks are small, we demonstrate that the strategies found by these local search methods is indeed very close to the best responses.

## 5.1 Social Networks

We consider two on-line social networks, namely the trust network of Epinions and the social network of the blog Slashdot. Since these networks have directed structures, to make the datasets consistent with PCBT concepts, we alter their structures to an undirected one by the procedure which is described in Figure 3. According to [25], there are two reasonable explanation for this modification. Firstly, in these datasets, the amount of reciprocated edges which go by different signs is insignificantly small(%0.0032 for Epinions and %0.0037 for slashdot). Therefore, the number of edges that are discarded are insignificant, too. Secondly, in balance theory, it is reasonable to expect reciprocated edges to be of the same sign since both positive and negative relations are mutual in nature.

There is a cost for hiding a relation (edge) in a network in PCBT game; therefore, we define a parameter to represent the costs of hiding, noted by $\alpha$, for each network, which is the upper bound on the costs of all edges in that particular network. The purpose is to investigate the relation between costs of hiding and both individual and overall pleasure. Furthermore, in all of the following experiments the costs of hiding edges for a graph of $\alpha$ upper bound is assigned in a random fashion based on a uniform probability distribution on the interval $(0, \alpha)$, i.e. it is equally likely for any value in this interval to be assigned to an edge.

We start by these networks as initial signed networks, iteratively. In each step, an $\alpha$ parameter is assigned to these networks and all nodes compute their current utility function, best response and the utility function achieved through applying the best strategy. However, nodes do not apply their computed best strategy to the network and the network does not change in any step. This process is repeated with incrementing $\alpha$ by 0.1 until reaching the value 30. We define the concept of *Price of Lying* (PoL) for a network to be sum of the best achieved pleasures (when players are able to lie) divided by former pleasures (without lie) to measure the overall potential improvement on pleasures of players. Thus,

$$PoL = \sum_{v \in V(G)} \frac{u_0(v)}{u_1(v)}$$

Where, $u_0(v)$ is the utility of node $v$ in the initial network without lying about any of its relations, and $u_1(v)$ is utility of node $v$ achieved through lying about its relations, and $V(G)$ is the set of nodes in the network. Note that we calculated $u_1$ locally, i.e. for each $v \in V(G)$, $u_1(v)$ is calculated when only $v$ can lie.

It can be observed in Figure 4 that with $\alpha$ increasing the *price of lying* decreases in value for both of the networks, meaning that lower quantity pleasures can be achieved by players because they are obliged to play more honestly due to high average costs of hiding relations. This observation leads us to draw a rather odd conclusion that being able to manipulate other players view of the game status helps players to gain better outcomes for players overall. In other

| k | $u \to v$ | $v \to u$ | $u \leftrightarrow v$ |
|---|---|---|---|
| 0 | + | + | + |
| 1 | + | $\phi$ | + |
| 2 | $\phi$ | + | + |
| 3 | - | - | - |
| 4 | - | $\phi$ | - |
| 5 | $\phi$ | - | - |
| 6 | + | - | $\phi$ |
| 7 | - | + | $\phi$ |

**Fig. 3.** Method proposed for obtaining an undirected signed network from a directed one. This works because our datasets have very small fraction of reciprocated edges and in the balance theory it is reasonable to expect that the reciprocated edges have same signs. +, - and $\phi$ represent positive,negative, and unknown relations between nodes u and v respectively [25].

words, lying can improve social welfare in the case of this particular setting. Note that the game does not advance in this scenario and we measure the potential improvement of pleasures for nodes, one by one, on the initial social network without making any modifications to it. Consequently, in the following section, we explore another scenario in which the games go on step by step, and nodes with respect to their turn, find their best strategy and apply it to their current graph until the game reaches equilibrium.

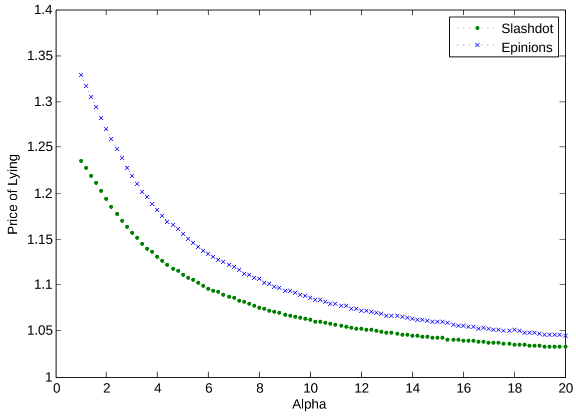



**Fig. 4.** Prices of lying calculated for two social networks of Epinions and Slashdot with different overall costs of hiding relations.

### 5.2 Randomly Generated Networks

To study the influence of lying on the social welfare of players involved, it is necessary that we study this game dynamically. Therefore, we devise a scenario in which a series of PCBT games is carried out in a set of networks and we analyze the relation between costs of lying about edges and the pleasure of the players.

In order to investigate the effects of the addition of the concept of lying to *Balance Theory* games, on the outcomes of these games, we define two parameters for each network. One to measure the costs of lying and another to measure the overall pleasure gained by players involved in the game. Again, we define $\alpha$ to be the upper bound on the costs of hiding the edges of the graph. Additionally, we define *normalized total pleasure*, denoted by $P$, for a game on a network as the sum of the pleasures of all vertices when they can lie about their relations, divided by the sum of pleasures of all vertices when they cannot lie about their relations in the underlying network.

As we want to investigate the dependency between the parameters $\alpha$ and normalized total pleasure in PCBT games network, we generate networks with different $\alpha$ parameters and simulate a PCBT games on each of these networks until they reach equilibrium; then, normalized total pleasure is calculated for each of these games. Iteratively, we randomly generate a network of 40 nodes and assign an $\alpha$ to it, which is incremented by a constant value in comparison to the graph of the previous iteration. Similar to the previous scenario, we assign costs to hiding edges in this network using a uniform probability distribution on the interval $(0, \alpha)$. Afterwards, three games is simulated on each of these graphs. The difference between these games is using disparate local search heuristics by players to explore possible strategies to find their best response. The computed normalized total pleasures for each combination of heuristic and $\alpha$ is presented in Figure 5. Although the outcomes of games with players using different heuristics to find their best response is not the same as each other, they are quite similar. This observation demonstrates that the local search method for finding best responses yields proper approximations for best responses.

Figure 5 shows an interesting property about the PCBT games. Lower overall costs for hiding relations (players playing less honestly as a consequence) leads to lower normalized total pleasure for a PCBT game. However, with penalties being insignificant in average, as opposed to many real world situations, normalized total pleasures of higher values are produced, which is why a decrease is noticed in the value of normalized total pleasure initially. This observation (players receiving better outcomes by playing more honestly) seems to be in direct contrast with the one from previous experiment scenario. This contradiction can be explained by the fact that in the first case only the effects of one player lying is studied where a signification improvement on pleasure is resulted because of other nodes playing honestly and the equivalence of a player's perception of the status of existing relations with the status of actual ones.

Thus, it can be inferred that if players in a PCBT game lie more about their relations, they achieve pleasures of less quantity overall, whereas, if they play or are obliged to play more honestly, an increase will follow in the pleasures and social welfare consequently.

## 6 Conclusions

In this paper, we proposed a prize collecting version of the model introduced in [25]. Our model is a network creation game for studying the evolution of signed networks which is based on the theory of structural balance. The utility of each player is defined as the notion of stability proposed in [25] minus the amount of pressure he feels from lying about some of his relations to others. We prove that with adding this new feature to the model, the Nash equilibrium graphs have different structural properties which are more natural than the previous model.

Our model can adopt useful extensions. Some of the natural extensions are listed below:

- Computing the best response in PCBT-Game is categorized as a Net Worth Maximization problem in the class of prize collecting problems [21]. Considering other categories such as Goemans-Williamson Minimization might be interesting.
- Szell shows that a vast majority of changes in the signed networks are due to the creation of new positive and negative links and not because of changing the sign of an edge [31]. Based on this fact, a model is introduced in [25] which needs theoretical verifications. Extensions of this model are also important.
- The asymmetric case where $C_u(v)$ is not necessarily equal to $C_v(u)$ is open to study. Our experimental evaluations show that this version does not converge in some cases.

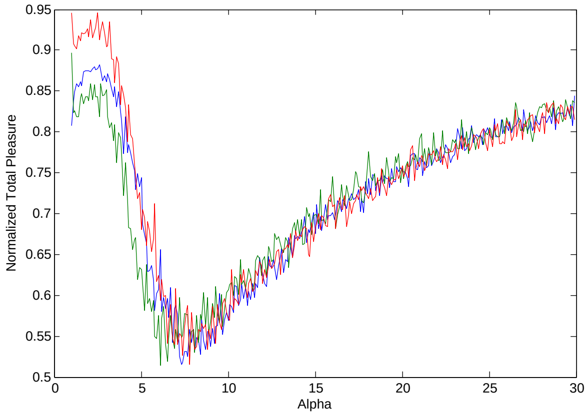



**Fig. 5.** Changes in the normalized total pleasure of networks with the upper bounds on costs increasing. Each curve represents the result obtained by applying a particular heuristic for finding best responses. This shows how the results tend to be similar to each other and a decent approximation for the best response.

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