On a Bounded Budget Network Creation Game

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ABSTRACT

We consider a network creation game in which, each player (vertex) has a limited budget to establish links to other players. In our model, each link has a unit cost and each agent tries to minimize its cost which is its local diameter or its total distance to other players in the (undirected) underlying graph of the created network. Two variants of the game are studied: in the MAX version, the cost incurred to a vertex is the maximum distance between that vertex and other vertices, and in the SUM version, the cost incurred to a vertex is the sum of distances between that vertex and other vertices. We prove that in both versions pure Nash equilibria exist, but the problem of finding the best response of a vertex is NP-hard. Next, we study the maximum possible diameter of an equilibrium graph with n vertices in various cases. For infinite numbers of n, we construct an equilibrium tree with diameter $\Theta(n)$ in the MAX version. Also, we prove that the diameter of any equilibrium tree is $O(\log n)$ in the SUM version and this bound is tight. When all vertices have unit budgets (i.e. can establish link to just one vertex), the diameter in both versions is $O(1)$. We give an example of equilibrium graph in MAX version, such that all vertices have positive budgets and yet the diameter is as large as $\Omega(\sqrt{\log n})$. This interesting result shows that the diameter does not decrease necessarily and may increase as the budgets are increased. For the SUM version, we prove that every equilibrium graph has diameter $2^{O(\sqrt{\log n})}$ when all vertices have positive budgets. Moreover, if the budget of every players is at least k, then every equilibrium graph with diameter more than 3 is k-connected.

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1. INTRODUCTION

In recent years, a lot of research has been conducted on network design problems, because of their importance in computer science and operations research. The aim in this line of research is usually to build a minimum cost network that satisfies certain properties. The most well studied problem in this area is, perhaps, the problem of finding the minimum cost spanning tree. The network structure is usually determined by a central authority. This is, however, in contrast to many real world situations such as social networks, where networks are formed in a distributed manner by selfish agents. Therefore, a novel game theoretic approach has also been proposed (see [6, 1, 5]), in which it is assumed that each agent has its own objective, and attempts to minimize the cost it incurs in the network, regardless of how its actions affect other agents.

Fabrikant et al. [6] first introduced this approach and took into account both the creation and the usage cost of the network. In their model, the players correspond to the vertices of the network graph, and every player aims at minimizing the sum of its shortest-path distances to other vertices plus the price she pays for building links (edges) to other players. After that, various network creation games were proposed (see [1, 5, 4, 3, 7]), which vary in the way players participate in network creation. In most of these games, there is a certain cost for building links, and the goal of each player is to minimize its maximum distance or total distance to other vertices.

Our work is motivated by the work of Laoutaris et al.[10]. In their model, every player has a specific budget for purchasing links. The objective of every player is to use its budget to establish some links to other vertices so as to minimize its maximum distance or total distance to other vertices in the resulting network. They focused on the case where all players have the same budget and all links cost are the same, so each player can establish a fixed number of links. In their model, links are directed and properties of the created directed graph is studied.

In this paper, we have considered an undirected variant of their model. In our model, once a link is established, both its endpoints can use it equally. This is a natural model in applications where the direction of links does not matter, for example, in computer networks. Although in our model links are undirected, each edge has just one owner and only one of its endpoints can be changed during the game. We also allow the players to have non-equal budgets.

1.1 The model and notation

Let n be a positive integer and $d_1, d_2, \ldots, d_n$ be nonnegative integers. A *bounded budget network creation game* with parameters $d_1, d_2, \ldots, d_n$, denoted by $(d_1, d_2, \cdots, d_n)$-BG, is the following game. There are n players and the strategy of player i is a subset $S_i \subseteq \{1, 2, \ldots, n\} \backslash \{i\}$ with $|S_i| = d_i$. We may build a directed graph G for every strategy profile $(S_1, \ldots, S_n)$ of this game, which has vertex set $V(G) = \{u_1, \ldots, u_n\}$, and for all i, j, (u_i, u_j) is an arc in G if $j \in S_i$. If (u_i, u_j) is an arc, then we say the arc (u_i, u_j) is *owned by* player i. As there is clear correspondence between the players and the vertices, we may sometime abuse notation and write statements like "vertex u_i owns the arc (u_i, u_j)," or "player i has an arc to vertex u_j." We think of the d_i as the *budget* available to player i. The *underlying graph* of G, which is an undirected graph obtained by ignoring the edge directions in G, is denoted by $U(G)$. If both arcs (u_i, u_j) and (u_j, u_i) are in G, then there is only one edge u_iu_j in $U(G)$ (see Fig. 1). In this case, the edge u_iu_j is called a *double edge*. In the following, whenever we refer to the distance between two vertices, we mean their distance in $U(G)$. The distance between two vertices u, v is denoted by $dist(u, v)$. For a directed or undirected graph G, the *diameter* of G is the maximum distance between any two vertices of G.

We define two models for the bounded budget network creation game, which differ in the definition of the cost function. In the *SUM* model, the cost of each player is the sum of its distances to other vertices, that is, for each vertex $u \in V(G)$,

$$c_{SUM}(u) = \sum_{v \in V(G)} dist(u, v)$$

while in the *MAX* model, the cost of each player is the maximum of its distances to other vertices, that is, for each vertex $u \in V(G)$,

$$c_{MAX}(u) = \max\{dist(u, v) : v \in V(G)\}$$

The value $c_{MAX}(u)$ is sometimes called the *local diameter* of u.

We say a player is playing its *best response* if it cannot decrease its cost by changing its strategy (while the other players' strategies are fixed), and we say a strategy profile is a *Nash Equilibrium (NE)* if all players are playing their best responses. If this happens, then the graph G is also said to be a *Nash Equilibrium graph*, or simply an *equilibrium graph* for $(d_1, d_2, \ldots, d_n)$-BG.

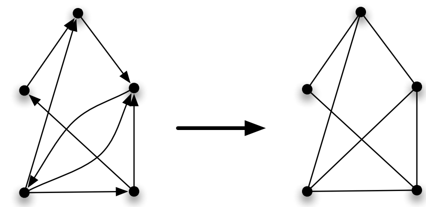

Figure 1: The illustration of $U(G)$

We have also studied the *Price of Anarchy (PoA)* and the *Price of Stability (PoS)* where the social utility function is the diameter of the graph. Price of Anarchy, introduced by Papadimitriou et al.[9], measures the effect of selfish agents on social utility, i.e. computes the worst ratio of the value of social utility function on every (pure) equilibrium to the optimal value. The best such ratio is called the Price of Stability. In this paper, the social utility function is the diameter of the created network.

1.2 Our results

In this paper, we study various properties of equilibrium graphs for bounded budget network creation game. First, in the next section, we prove that for every nonnegative sequence $d_1, \ldots, d_n$, $(d_1, d_2, \cdots, d_n)$-BG has a Nash equilibrium in both models. Next, we turn our attention to the diameter of equilibrium graphs. Our focus in this part are equilibria that have maximum diameters, which is related to the concept of price of anarchy when the social utility is the diameter of the created graph. We consider two special cases in Section 3, and find tight bounds for the maximum diameter. The two cases are unit budgets (in which $d_i = 1$, for every i) and trees (in which $d_1 + d_2 + \cdots + d_n = n - 1$). For the former, we prove that the diameter is always bounded above by a constant, and for the latter, we prove a $\Theta(n)$ bound for the MAX version and a $\Theta(\log n)$ bound for the SUM version. Then, in Section 4, we consider a more general case in which $d_i \geq 1$ for all $1 \leq i \leq n$, and obtain an upper bound $2^{O(\sqrt{\log n})}$ for the SUM version, and a lower bound $\Omega(\sqrt{\log n})$ for the MAX version.

The latter result disproves an intuitive guess that increasing the budgets, i.e the d_i's, decreases the diameter of equilibrium graphs: while the diameter is $O(1)$ for the unit degree case it could be as large as $\Omega(\sqrt{\log n})$ for larger values of d_i's in the MAX version. We also prove that in the SUM version, if $d_i \geq k$ for all i, then every equilibrium graph with diameter more than 3, is k-connected. We conclude with discussion of our results and suggesting some interesting open problems.

Table 1: The results of this paper on diameter of the equilibrium graphs

	MAX	SUM
Tree	$\Omega(n)$	$\Theta(\log(n))$
Unit Budget	$O(1)$	$O(1)$
General	$\Omega(\sqrt{\log(n)})$	$2^{O(\sqrt{\log n})}$

2. EXISTENCE OF EQUILIBRIA

Before proving the main result of this section, we show that computing best response is an intractable problem.

Theorem 1. *The problem of finding the best response in both MAX and SUM models of $(d_1, \cdots, d_n)$-BG is NP-Hard.*

Proof. We can reduce the k-center problem [8] to the problem of finding the best response in the MAX version of the game. In the k-center problem, a graph G is given and the aim is to find a subset C of k vertices of G so as to minimize the maximum distance from a vertex to its nearest neighbor in C, i.e. $\max_{v\in V(G)} \min_{c\in C} dist(c, v)$. Assume that we are given an undirected graph H, and we are supposed to find its k-center. Add a vertex $n+1$ to H and define $d_{n+1} = k$. Consider a directed graph G such that $U(G) = H$. Now compute the best response for the $(n+1)$'th player in MAX version in response to G. This essentially finds a subset of k vertices of G whose maximum distance to the remaining vertices of G is minimized, which is clearly a k-center for G.

Similarly, we can reduce the k-median problem [11] to find the best response in the SUM version of the game. □

In this section, we prove that for every nonnegative $d_1, d_2, \cdots, d_n$, Nash equilibria exist for both MAX and SUM versions. First, we prove a sufficient condition for each vertex to play its best response, and then prove the main theorem by considering several cases. The diameter of the equilibrium constructed in this theorem is $O(1)$ which proves that the price of stability is $O(1)$.

Lemma 1. *Let u be a vertex. If $c_{MAX}(u) \le 2$ and u is not an endpoint of any double edge or $c_{MAX}(u) = 1$ then u plays its best response in both MAX and SUM models.*

Proof. If $c_{MAX}(u) = 1$, then we are done. Otherwise, let V^- be the set of vertices that have an arc to u and V^+ be the set of vertices that have an arc from u. Since u is not an endpoint of any double edge, $V^+ \cap V^- = \emptyset$. It is easy to verify that no matter how u plays, it always has distance one to at most $|V^+|+|V^-|$ vertices, and distance at least two to the rest of the vertices. Therefore, regardless of how u plays, its cost in MAX model will be at least 2, and its cost in SUM model will be at least $2(n-1-|V^-|-|V^+|)+|V^+|+|V^-|$. Therefore, u is already playing its best response. □

We are now ready to prove the main theorem of this section.

Theorem 2. *For every nonnegative $d_1, d_2, \cdots, d_n$, Nash equilibria exists for both MAX and SUM versions of $(d_1, \ldots, d_n)$-BG.*

Proof. The proof is constructive. We consider several cases, and prove it separately for each case. Without loss of generality, assume that $d_1 \le d_2 \le \cdots \le d_n$.

Let $D = d_1 + d_2 + \cdots + d_n$. If $D < n-1$ then the obtained graph, $U(G)$, is always disconnected and both MAX and SUM costs of every vertex are ∞. Therefore, for all vertices, every strategy is a best response. So, assume that $D \ge n-1$. Let z be the number of players with zero budget, so we have $d_1 = \cdots = d_z = 0 < d_{z+1}$. There are two cases to consider:

Case 1: $d_n \ge z$

We provide an algorithm to build a graph G, such that all of its vertices satisfy the conditions of Lemma 1. G has vertex set $\{u_1, \ldots, u_n\}$ and is initially empty. We add the arcs (u_n, u_1), (u_n, u_2), $\ldots$,(u_n, u_{d_n}) and then the arcs (u_{d_n+1}, u_n),(u_{d_n+2}, u_n),$\ldots$,(u_{n-1}, u_n) to G. Note that G has diameter 2 at this point, but there might be vertices whose outdegrees are less than their budgets. If u_i is such a vertex, add arcs from u_i to arbitrary vertices until its budget is consumed. This operation clearly does not increase the diameter, but this may result in double edges. For every double edge uv such that u has local diameter two and there exists a vertex w not adjacent to u, replace the arc (u, v) by (u, w). This can be done only a finite number of times, since after every replacement the number of double edges decreases. It is easy to see that the vertices of the obtained graph have the properties of Lemma 1 and thus this graph is a NE.

Case 2: $d_n < z$

As in Case 1, we build a graph G that is a Nash equilibrium, but the proof is more involved in this case. Let $t > z$ be the largest index with $d_n + d_{n-1} + \cdots + d_t \ge z + n - t$. Such value of t exists, as for $t = z+1$, we have $d_n + d_{n-1} + \cdots + d_{z+1} = D \ge n-1 = z+n-t$. Let $A = \{v_1, v_2, \cdots, v_z\}$, $B = \{v_{z+1}, v_{z+2}, \cdots, v_t\}$ and $C = \{v_{t+1}, v_{t+2}, \cdots, v_{n-1}\}$.

We start with an empty graph G and add arcs in four steps until the budgets of vertices are consumed (See Fig. 2)

1. An arc from every vertex in $B \cup C$ to v_n (dotted arcs in Fig. 2).

2. Arcs from $\{v_t\} \cup C \cup \{v_n\}$ to A. First, d_n arcs from v_n to the first d_n vertices of A then $d_{n-1}-1$ arcs from v_{n-1} to the next $d_{n-1}-1$ vertices of A and so on, until every vertex in A receives *exactly one* arc (dashed arcs in Fig. 2).

3. Arcs from B to C. For every vertex u in B that has remaining budget, we add arcs from u to vertices in C in reverse order, i.e. $v_{n-1}, v_{n-2}, \cdots$ (gray arcs in Fig. 2).

4. Arcs from B to A. For every vertex u in B that still has remaining budget, we add arcs from u to vertices in A in order, i.e. $v_1, v_2, \cdots$. So, every vertex in B is only adjacent to neighboring vertices of v_n in A because for every $z < i \le t$, we have $d_i \le d_n$.(black arcs in Fig. 2).

We now prove that every vertex is playing its best response in this graph. Vertices in A are obviously playing their best strategies as their budgets are zero. It is easy to verify that we are not creating a double edge in our construction. Since v_n has local diameter two, it plays its best response by Lemma 1. Every arc from a vertex $u \in C$ is either connected to v_n or to some vertex in $v \in A$. The latter cannot be changed, as changing it would disconnect v from G and increases the cost of u. It is also easy to verify that u is better off staying connected to v_n.

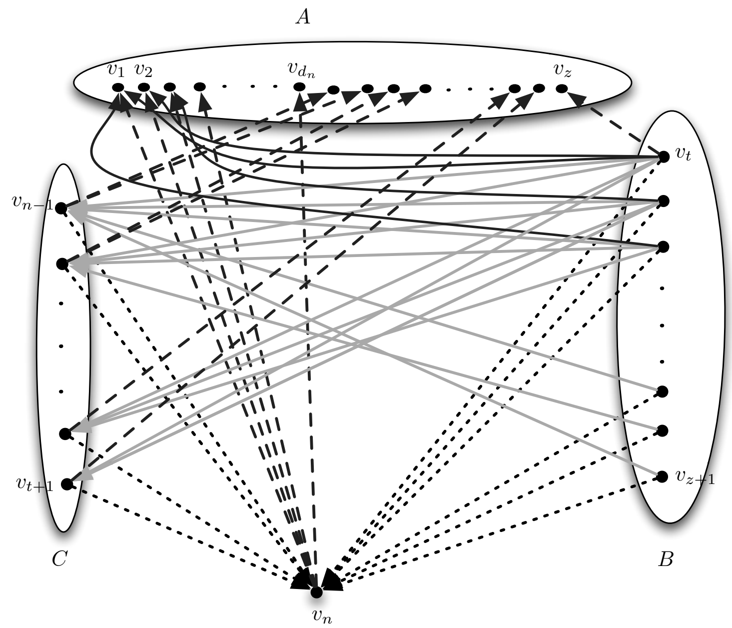

Figure 2: Case 2 of theorem 2

At last, consider a vertex u in B. If u creates arcs in step 4, then it has arcs to all vertices in C (step 3) and therefore, it has diameter two, since every vertex in A is a neighbour of v_n or one vertex in C. Thus in this case, vertex u satisfies the conditions of Lemma 1 and it plays its best response. Otherwise, u has local diameter three. The vertex u must have $K < |C|$ available budget in step 3. First, it is clear that u does not change its arc to v_n. Furthermore, since for any vertex $w \in C$ there is a vertex $w' \in A$ such that w' is only adjacent to w in $U(G)$, vertex u can not make its local diameter less than 3. Thus in this case, it plays its best response in the MAX version. Also, in the SUM version, it is easy to verify that its best strategy is to connect to the most influential vertices, i.e. $v_n, v_{n-1}, \cdots, v_{n-K}$. $\square$

3. SPECIAL CASES

In this section, we find tight bounds for price of anarchy in two special cases. First we consider a situation where every player has unit budget and prove that the diameter of equilibrium graphs is bounded by a constant. Next, we consider the case where the equilibrium graphs are always trees, and find different bounds for MAX and SUM variants.

Note: we only consider connected equilibrium graphs. Any graph with more than d components, where $d = \max\{d_1, d_2, \cdots, d_n\}$, is a Nash equilibrium as no vertex can make the graph connected and the maximum and total distance of every player is always infinity. One way to solve this is to take the number of connected components into account for cost functions and, therefore, encourage players to reduce the number of connected components even though that may not reduce their local diameter.

3.1 Unit Budget Case

One special case of the problem is when all d_i's are one. In this case, we prove that all equilibrium graphs have diameter $O(1)$. The proof is left to the journal version due to space shortage.

THEOREM 3. *In $(1, 1, \cdots, 1)$-BG all the equilibrium graphs in both MAX and SUM versions have $O(1)$ diameters.*

3.2 Trees

If $d_1 + d_2 + \cdots + d_n = n - 1$, then it can be easily seen that every equilibrium graph is a tree. From now on, we use the notion *Tree-BG* to indicate the instances of bounded budget network creation games that have $\sum_{i=1}^{n} d_i = n - 1$. In this section, we study the diameter of connected equilibrium graphs of Tree-BG in both MAX and SUM models. We prove that in the MAX model, there exists equilibrium graphs with diameter $\Omega(n)$, while in the SUM model, equilibrium graphs always have diameter $O(\log n)$, and this bound is tight.

THEOREM 4. *In the MAX model, there are Tree-BG instances that have equilibrium graphs with diameter $\Omega(n)$.*

PROOF. Let k be a positive integer, and let $n = 3k + 1$, $X = \{x_1, x_2, \cdots, x_k\}$, $Y = \{y_1, y_2, \cdots, y_k\}$, and $Z = \{z_1, z_2, \cdots, z_k\}$. Also, let $d_1 = \cdots = d_4 = 0$, $d_5 = d_6 = d_7 = 2$, and $d_8 = d_9 = \cdots = d_n = 1$. Let G be a graph with vertex set $X \cup Y \cup Z \cup \{w\}$ and with set of arcs

$\{(x_1, x_2), \ldots, (x_{k-1}, x_k)\} \cup \{(y_1, y_2), \ldots, (y_{k-1}, y_k)\} \cup$
$\{(z_1, z_2), \ldots, (z_{k-1}, z_k)\} \cup \{(x_1, w), (y_1, w), (z_1, w)\}$

We claim that for all $1 \leq i \leq k$, x_i is playing its best response. The proof for y_i's and z_i's are similar. If $i > 1$, then x_i has unit budget and currently has an arc to x_{i+1}. If it changes its arc to (x_i, x_j) for some $j > i + 1$, then its local diameter doesn't decrease. If it changes to any other arc, then the graph gets disconnected, and x_i will have infinite local diameter.

If $i = 1$, then x_1 should choose a vertex from each of the two disjoint paths $x_2x_3 \cdots x_k$ and $z_kz_{k-1}z_1wy_1y_2 \cdots y_k$, and establish links to these two vertices otherwise the graph will be disconnected. Its best response is obviously to choose the middle of the second path (which is w) and an arbitrary vertex in the first path.

□

In the next theorem, we will show that the diameters of equilibrium graphs in the SUM model are much smaller.

THEOREM 5. *In SUM model, all equilibrium graphs of Tree-BG have diameter $O(\log(n))$.*

PROOF. Let G be an equilibrium graph with diameter d, and let $P = v_0v_1 \cdots v_d$ be its longest path. Trivially at least half of the arcs of P are in the same direction along P. By symmetry, we may assume that these are the arcs $(v_{i_1}, v_{i_1+1}), (v_{i_2}, v_{i_2+1}), \ldots, (v_{i_{\lceil d/2\rceil}}, v_{i_{\lceil d/2\rceil}+1})$. Let A_i be the set of vertices that are connected to P through v_i (including v_i), and let $a_i = |A_i|$. Notice that $a_0 = a_d = 1$ as P is the longest path in G. See Fig. 3 for an example.

For $1 \le j < \lceil d/2\rceil$, if v_{i_j} changes its arc from (v_{i_j}, v_{i_j+1}) to (v_{i_j}, v_{i_j+2}), then its distance to vertices in A_{i_j+1} increases by one, and its distance to vertices in A_k, $k > i_j + 1$, decreases by one. Since v_{i_j} is playing its best response,

$$a_{i_j+1} \ge \sum_{k=i_j+2}^{d} a_k \ge \sum_{k=j+1}^{\lceil \frac{d}{2}\rceil} a_{i_k+1}$$

Since $v_j \in A_j$, we have $a_j \ge 1$.

$$\begin{array}{lcl} a_{(i_{\lceil \frac{d}{2}\rceil}+1)} & \ge & 1 \\ a_{(i_{\lceil \frac{d}{2}\rceil-1}+1)} & \ge & a_{(i_{\lceil \frac{d}{2}\rceil}+1)} \\ a_{(i_{\lceil \frac{d}{2}\rceil-2}+1)} & \ge & a_{(i_{\lceil \frac{d}{2}\rceil-1}+1)} + a_{(i_{\lceil \frac{d}{2}\rceil}+1)} \\ & \ldots & \\ a_{(i_2+1)} & \ge & a_{i_3+1} + \cdots + a_{(i_{\lceil \frac{d}{2}\rceil}+1)} \\ a_{(i_1+1)} & \ge & a_{i_2+1} + a_{i_3+1} + \cdots + a_{(i_{\lceil \frac{d}{2}\rceil}+1)} \end{array}$$

We can prove by induction that $a_{i_j+1} \ge 2^{\lceil \frac{d}{2}\rceil - j - 1}$ for $1 \le j < \lceil \frac{d}{2}\rceil$. Therefore,

$$\begin{array}{lc} a_{(i_1+1)} + a_{(i_2+1)} + \cdots + a_{(i_{\lceil \frac{d}{2}\rceil}+1)} & \ge \\ 2^{\lceil \frac{d}{2}\rceil-2} + 2^{\lceil \frac{d}{2}\rceil-3} + \cdots + 2^1 + 2^0 + 2^0 & = \\ 2^{\lceil \frac{d}{2}\rceil-1} & \end{array}$$

On the other hand, since all vertices appear in one of the sets A_i, we have $a_1 + a_2 + \cdots + a_d = n - 1$. Thus,

$$n - 1 = a_1 + a_2 + \cdots + a_d \ge \sum_{j=1}^{\lceil \frac{d}{2}\rceil} a_{(i_j+1)} \ge 2^{\lceil \frac{d}{2}\rceil-1},$$

Therefore, $d = O(\log n)$. □

The bound $O(\log n)$ is tight as there exist instances with diameter $\Omega(\log n)$.

THEOREM 6. *For infinitely many n, there exists an equilibrium graph for Tree-BG in the SUM model with diameter $\Omega(\log(n))$.*

PROOF. Let k be a positive integer, and let $n = 2^{k+1} - 1$, $d_1 = d_2 = \cdots = d_{2^k-1} = 2, d_{2^k} = d_{2^k+1} = \cdots = d_n = 0$, Consider a balanced binary tree on n vertices in which vertex i $(1 \le i < n/2)$ has arcs to vertices $2i$ and $2i + 1$. For each i, let T_i be the tree rooted at vertex i. For each $i < 2^k$, vertex i must have an arc to a vertex in T_{2i} and to a vertex in T_{2i+1} in order to keep the graph connected. Observe that for every j, vertex j has less total distance to vertices in T_j than any other vertex in T_j, and so all vertices are playing their best responses. The diameter of this equilibrium graph is $2(\log(n + 1) - 1) = \Theta(\log(n))$. □

4. GENERAL CASE

In this section, we assume that all players have positive budgets i.e. for each $1 \le i \le n$, $d_i \ge 1$. It appears intuitive that increasing the budgets (i.e. d_i's) would decrease the diameter. This is, however, not true and we prove an $\Omega(\sqrt{\log(n)})$ lower bound for the price of anarchy in the MAX version. We also prove that the diameter of an equilibrium graph in the SUM version is $2^{O(\sqrt{\log n})}$.

4.1 Upper bound for SUM

In this subsection we consider the SUM model only, and prove that for any NE graph the diameter is $2^{O(\sqrt{\log n})}$. The proof follows the line of proof of Theorem 9 of [2], but the first step is more involved. Specifically, the proof of the following proposition, which is somewhat easy in the model defined in [2], is much harder in our model.

PROPOSITION 1. *Let u be a vertex of an NE graph G and r be a positive integer. Assume that the subgraph of $U(G)$ induced by the set of vertices whose distance from u is at most r, is a tree. Then we have $r = O(\log |V(G)|)$.*

We will work with weighted graphs in this subsection. Let G be a weighted graph, that is, every vertex u has a weight $w(u)$, which is a positive integer. For every vertex u, the cost of u is defined as

$$c(u) = \sum_{v \in V} w(v) dist(u, v).$$

Note that if all vertices have unit weight, then this reduces to our unweighted model. We say that G is a *weak Nash equilibrium* (abbreviated wNE) if for every arc $(u, v) \in E$ and $x \in V$ with $(u, x) \notin E$, the cost of u does not decrease if we replace the arc (u, v) with (u, x). For a vertex u and a nonnegative integer r, define

$$B_r(u) = \{v : dist(u, v) \le r\}.$$

For $A \subseteq V$ define

$$w(A) = w_A = w(G[A]) = w_{G[A]} = \sum_{a \in A} w(a),$$

where $G[A]$ denotes the directed subgraph of G induced by A.

Clearly a Nash equilibrium graph is also a weak Nash equilibrium graph, and thus it is enough to show that the diameter of any wNE graph is $2^{O(\sqrt{\log n})}$.

Using the defined notation, we will prove the following generalization of Proposition 1:

LEMMA 2. *Let G be wNE, $u \in V$ and $r > 0$. Assume that $U(G[B_r(u)])$ is a tree T. Then we have $r = O(\log |w_G|)$.*

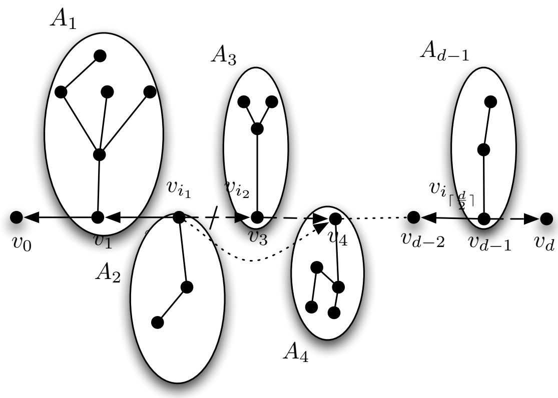

Figure 3: The proof of theorem 5

Note that if every vertex $v \in V(T)$ has at least two children then $r = O(\log |V(T)|) = O(\log |w_T|)$. Hence the problematic vertices are those with zero or one child. A vertex with degree 1 is called a *leaf*. Thus every vertex in T with no child is a leaf. It turns out that one should distinguish between two types of leaves: a *poor* leaf is a leaf with out-degree zero, and a *rich* leaf is a leaf with out-degree one. The poor leaves cause the most trouble and they are the reason for introducing the weights.

Let $l \in V$ be a poor leaf in G and $(u, l) \in E$. Let $G_0 = (V_0, E_0, w_0)$ be a weighted graph with $V_0 = V - \{l\}$, $E_0 = E - \{(u, l)\}$, $w_0(v) = w(v)$ for $v \neq u$ and $w_0(u) = w(u) + w(l)$. Then it can be verified that if G is a wNE then so is G_0. We say that G_0 is obtained by *folding* the poor leaf l. The proof of the following lemma is left to the journal version of this paper due to space shortage.

Lemma 3. *Let G be a wNE and T be an induced subtree of $U(G)$. Let $r \in V(T)$ be such that if we choose r as the root of T, then every edge of T is oriented away from r. In other words, if u, v are vertices in T and (u, v) is an arc in G then u is the parent of v in T. Then the depth of T is at most* $\log w_T + 1$.

Remark. Note that if the conditions of the above lemma hold, then one can fold the whole subtree T into the vertex r. Moreover, folding this subtree does not decrease the diameter of G significantly. That is, if G is wNE and we perform a sequence of subtree folds on it until we reach a new digraph G' with no poor leaves, then G' is also wNE and $\text{diam}(G) = \text{diam}(G') + O(\log w(G))$.

From now on, we will assume that the weak Nash equilibrium we are studying has no poor leaves (the diameter would be the same, modulo an $O(\log w(G))$ term). Handling rich leaves is easy, as shown by the following lemma (whose proof can be found in the journal version of this paper).

Lemma 4. *Let G be wNE. Then the distance between any two rich leaves is at most 2.*

To handle the vertices of degree 2 (which have one child) we use the following lemma, whose proof can be found in the journal version as well.

Lemma 5. *Let G be wNE and P be a path in $U(G)$ such that for every two vertices $u, v \in V(P)$, the shortest (u, v)-path in P is the unique shortest (u, v)-path (which implies, in particular, that P is an induced subgraph). Then the number of edges $\{u, v\} \in E(P)$ such that both u, v have degree 2 is* $O(\log w_P)$.

Proof of Lemma 2. By the remark after Lemma 3 we may assume that G has no poor leaves. For each edge $\{u, v\} \in E(P)$ such that both u and v have degree 2, we contract that edge, and repeat until no such edge exists. By Lemma 5, the depth changes by at most $O(\log w_G)$. By Lemma 4, there is at most one vertex that has children who are leaves. Hence the depth of the tree is $O(\log |V(T)|)$. Consequently, the depth of T is $O(\log w_G)$. $\square$

The rest of the proof is almost identical to the proof of Theorem 9 of [2]. In the following we will assume that the graphs are unweighted (equivalently, all vertices have unit weights). Note that in this case $w_G = n$.

Lemma 6. *Let G be wNE. Given any vertex u, there is an arc (x, y) with $dist(x, u) = O(\log n)$ and whose removal increases the cost of x by at most $O(n \log n)$.*

Hence for some constants $a, b > 0$, if G is an wNE then for any $u \in V$, there is an arc (x, y) with $dist(x, u) \leq a \log n$ and whose removal increases the cost of x by at most $bn \log n$. The proof of the following can be found in the journal version of this paper.

Corollary 1. *In any wNE the addition of any arc (u, v) decreases the cost of u by at most $(a + b + 1)n \log n$.*

Theorem 7. *The diameter of any wNE is $2^{O(\sqrt{\log n})}$.*

4.2 Lower bound for MAX

In this section, we prove that for some positive d_i's there exist equilibrium graphs for MAX model with diameter $\Omega(\sqrt{\log n})$.

For an undirected graph U, vertex $u \in V(U)$ and subset $A \subseteq V(U)$, the distance between u and A is defined as $dist(u, A) = \min\{dist(u, a) : a \in A\}$.

Lemma 7. *Let U be an undirected graph with diameter k and maximum degree Δ with the following properties:*

1. *All vertices have local diameter k.*

2. $\Delta^k - 1 < n(\Delta - 1)$.

Then every G with no double edge, with $U = U(G)$ is a Nash equilibrium for the MAX model.

Proof. Assume for the sake of contradiction that v is a vertex that is not playing its best response. Let A be the set of neighbors of v if it had changed its strategy and played its best response. As v has degree at most Δ, we have $|A| \leq \Delta$. By property (1) the local diameter of v is exactly k before changing its strategy.

Claim. There exists a vertex u, different from v, with $dist(u, A) > k - 2$.

Proof. There are at most $|A|\Delta$ vertices whose distance from A is exactly 1. Similarly, there are at most $|A|\Delta^2$ vertices with distance exactly 2 from A. Continuing in the same way, we find that there are at most $|A|\Delta^{k-2}$ vertices with distance exactly $k-2$ from A. If there is no $u \neq v$ with $dist(u, A) > k - 2$, then we must have

$$n \leq 1 + |A| + |A|\Delta + \cdots + |A|\Delta^{k-2} \leq 1 + \Delta + \Delta^2 + \cdots + \Delta^{k-1} = \frac{\Delta^k - 1}{\Delta - 1},$$

which contradicts the property (2). □

After v changes its strategy so that its neighborhood becomes A, its distance to u becomes at least k, which is a contradiction.

Lemma 8. *For every integers $t, k > 3$ satisfying $1 + 2^k < 2t$, there exists an undirected graph U with t^k vertices, minimum degree at least 2, and diameter k, such that every G with no double edge and $U = U(G)$ is a Nash equilibrium for the MAX model.*

Proof. Let $V(U) = \{1, 2, \ldots, t\}^k$ with $(a_1, a_2, \ldots, a_k)$ adjacent to $(b_1, b_2, \ldots, b_k)$ if at least one of the following happens:

1. $a_i = b_{i+1}$ for all $1 \leq i \leq k - 1$,

2. $b_i = a_{i+1}$ for all $1 \leq i \leq k - 1$.

Then U has minimum degree $2t - 2$, maximum degree $2t$ and t^k vertices. The local diameter of every vertex is k: for an arbitrary $(a_1, \ldots, a_k) \in V(U)$ choose $b_1, \ldots, b_k \notin \{a_1, \ldots, a_k\}$. Then it is easy to check that the distance between $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ is k. The condition $\Delta^k - 1 < n(\Delta - 1)$ of the previous Lemma follows from $1 + 2^k < 2t$ and a little calculation. □

Theorem 8. *For infinitely many n, there exists an equilibrium graph with positive d_i's for the MAX model with diameter $\sqrt{\log_2 n}$.*

Proof. Let $k > 3$ and $t = 2^k$. Using the previous theorem, we find an undirected graph U with $n = (2^k)^k = 2^{k^2}$ vertices and diameter $k = \sqrt{\log_2 n}$. Now, let G be a directed graph with $U(G) = U$ and such that the outgoing degree of all vertices of G is at least 1. Such a G can always be found as the minimum degree of U is larger than 1. Then G satisfies the conditions of the theorem. □

4.3 K-Connectivity

One of the most important issues in designing stable networks is the connectivity of the built network. In this section, we find a direct connection between the budget limits and the connectivity of the equilibrium graph, which shows that we can guarantee stronger connectivity for our network when all players have enough budgets. The proof is left to the journal version due to space shortage.

Theorem 9. *Suppose that G is an equilibrium graph for $(d_1, d_2, \ldots, d_n)$-BG in SUM version and $d_i \geq k$ for all $1 \leq i \leq n$. If G has diameter greater than 3, then it is k-vertex connected.*

5. CONCLUSION

In this paper, we analyzed the diameter of equilibrium graphs in network creation games where every player has a specific budget for the number of vertices that it can establish links to. We found tight bounds for two special cases, trees and unit budget. For the case where all players have positive budget, we proved a non-trivial lower bound for the MAX version and upper bound for SUM version.

Improving these bounds for both versions are interesting problems to work on. Considering other special cases (e.g. the case where $d_i = c$ for some constant $c \geq 2$) is also a good problem to work on. We have tried several examples and it appears that in the positive budgets case, the diameter of every SUM equilibrium is bounded by a constant. Either proving that this is correct, or finding a counter-example is another interesting open problem. Last but not least, the convergence rate of the game is another interesting parameter to study. That is, to determine how quickly the game converges to an equilibrium, if at each step, one player is chosen and plays its best response.

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7. REFERENCES

[1] S. Albers, S. Eilts, E. Even-Dar, Y. Mansour, and L. Roditty, On Nash equilibria for a network creation game, in Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), Miami, Florida, 2006, pp. 89-98.

[2] N.Alon, E.D.Demaine, M.Hajiaghayi, T.Leighton, Basic Network Creation Games, in Proceeding of the 22rd ACM Symposium on Parallelism in Algorithms and Architectures(SPAA),2010, pp. 106–113.

[3] N. Andelman, M. Feldman, and Y. Mansour, Strong price of anarchy, in Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), 2007, pp. 189-198.

[4] J. Corbo and D. Parkes, The price of selfish behavior in bilateral network formation, in Proceedings of the 24th Annual ACM Symposium on Principles of Distributed Computing (PODC), Las Vegas, Nevada, 2005, pp. 99-107.

[5] E.D. Demaine, M.Hajiaghayi, H.Mahini, and M.Zadimoghaddam. The Price of Anarchy in Network Creation Games. In Proceedings of the 26th

Annual ACM Symposium on Principles of Distributed Computing (PODC), Portland, Oregon, 2007, pp. 292-298.

[6] A. Fabrikant, A. Luthra, E. Maneva, C. H. Papadimitriou, and S. Shenker, On a network creation game, in Proceedings of the 22nd Annual Symposium on Principles of Distributed Computing (PODC), Boston, Massachusetts, 2003, pp. 347-351.

[7] Y. Halevi and Y. Mansour, A network creation game with nonuniform interests, in Proceedings of the 3rd International Workshop on Internet and Network Economics (WINE), vol. 4858 of Lecture Notes in Computer Science, San Diego, CA, 2007, pp. 287-292.

[8] W. L.Hsu and G.L.Nemhauser, Easy and hard bottleneck location problems, Disc. Appl. Math. 1, 209-216,1979.

[9] E.Koutsoupias and C.H. Papadimitriou, Worst-case equilibria, Computer Science Review, 1999, pp. 65-69.

[10] N. Laoutaris, L. J. Poplawski, R. Rajaraman, R. Sundaram, and S.-H. Teng, Bounded budget connection (BBC) games or how to make friends and influence people, on a budget, in Proceedings of the 27th ACM Symposium on Principles of Distributed Computing (PODC), 2008, pp. 165-174.

[11] J.H.Lin and J.S.Vitter, ε-approximations with minimum packing constraint violation, Proc. 24th Ann. ACM Symp. on Theory of Comp., ACM, 771-782, 1992.